THE UNIVERSITY OF ALBERTA

"INVESTIGATION INTO THE PROPERTIES OF A SLOW-WAVE STRUCTURE CONSISTING OF A CHAIN OF COUPLED CAVITIES"

by

© AHMAD SHAMALY

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFULMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF ELECTRICAL ENGINEERING

EDMONTON, ALBERTA

FALL, 1969

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "INVESTIGATION INTO THE PROPERTIES OF A SLOW-WAVE STRUCTURE CONSISTING OF A CHAIN OF COUPLED CAVITIES", submitted by AHMAD SHAMALY in partial fulfilment of the requirements for the degree of MASTER OF SCIENCE.

ABSTRACT

The behavior of standing wave accelerator tanks, made up of a chain of resonant cavities, is investigated using a coupled resonator model. The behavior of the dispersion curve in the vicinity of the operating frequency of the structure is determined. It is shown that the shape of the dispersion curve at the π -mode is greatly affected by the presence of losses in the cavities. A comparison is made between a coupled cavity chain and a dielectrically loaded slow wave structure.

ACKNOWLEDGMENT

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LIST OF SYMBOLS

R	Resistance of a cavity
L	Inductance
M	Mutual Inductance
C	Capacitance
I_n	Current in cavity 'n' (Phasor), or current input to the n^{th} section
N	Number of periodic sections
Z	Impedence
ϕ	General phase shift per section
ψ	Real phase shift per section
α	Attenuation per section
S	Unitary transformation
$\lambda_{1,2}$	Eigen-values of the transmission MATRIX
T	Transmission matrix of a periodic section
ℓ	Mode number
j	is the $\sqrt{-1}$
k	Coupling coefficient
Q_i	the Q of the i^{th} type cavity
W_0	Resonant frequency of the singly periodic structure cavities
$W_{1,2}$	Resonant frequencies of the two types of cavities in a doubly periodic structure
P	is the odd cavities label in a doubly periodic structure.
ε_i	Permittivity of an i^{th} dielectric region

μ	permeability
X, Y, W	Planes of symmetry
x_1, x_2, x_3, x_4	Components of the eigen vectors
A,B,C,D,	Elements of the "T" MATRIX
A_n, B_n, C_n, D_n	Elements of the $(T)^n$ MATRIX
Z_N	Terminating impedence of the N section
V_n	Voltage at the input of the n^{th} section
W	angular frequency
a	radius of the circular cylindrical wave guide
E_r, E_z	Components of the electric field in circular cylindrical coordinate
$H\phi$	component of the magnetic field
S_1	First root of the order zero Bessel function of the first kind
J_0, J_1	Bessel functions of the first kind of order zero and one respectively
$Z_T(z)$	Total impedence at an interface as a function of the z coordinate
Z_i	Characteristic impedence of a dielectric region characterized by ε_i
Z_a, Z_b	Impedences of the "T" circuit for one region
β_i	Phase shift in an isolated dielectric region
T_1, T_2	Transformation matrices for half sections in a doubly periodic structure

L	Length of a wave guide piece
θ_i	Phase shift in an isolated dielectric region characterized by ε_i
G	Amplitude of forward wave in dielectric
H	Amplitude of backward wave in dielectric
g	Length of air region in one periodic section
h	Length of dielectric region in one periodic section

INTRODUCTION

A particle beam can only interact continuously with a wave if they have approximately the same velocity. If the velocities are widely different, the particle will experience a high frequency field which will result in no average change in the particle energy.

A particle travelling slightly more slowly than the wave, and which enters the field at the beginning of the accelerating half-cycle, will initially acquire energy from the wave and then travel along with it. The allowable initial discrepancy in velocity is a function of the maximum accelerating field.

Slater (1) has shown that for a wave travelling at the velocity of light, continuous interaction is still possible, because although the particle must continuously slip back in phase, it will remain in the accelerating field indefinitely if the field is high enough.

West (2) has given a curve, derived from Slater, relating the intensity of the accelerating wave to the required initial particle velocity if capture is to take place.

An electron travelling at a velocity close to the velocity of light will interact therefore with a wave whose phase velocity is equal to the velocity of light. This is the principle employed in many electron accelerators.

In a circular cylindrical wave guide, the phase velocity is always greater than the velocity of light. To use the wave guide as an accelerator, therefore, slowing of the electromagnetic wave must take place. This may be accomplished by inserting metalic IRISES or dielectric disks inside the wave guide. Where the velocity of the accelerating wave must be constant, the loading elements must be placed in a periodic manner. For any periodic structure the field may be analyzed into space harmonics, which in general have different phase velocities, so that a particle beam will only interact with one of them.

In a resonant accelerator it is possible to obtain interaction of the beam with two space harmonics by using π-mode operation. This raises the accelerating efficiency but due to the zero group velocity, the mode separation is normally very small.

Walker and West (3) have shown, however, that in a wave guide loaded with dielectric disks, good mode separation at the π-mode may be obtained by arranging that the first and second pass bands of the structure are confluent.

A slow wave structure may also be constructed by coupling resonant cavities together in an identical manner.

It has been shown by Nagle and Knapp (4) that a chain of cavities can be used to accelerate a beam using the π-mode by allowing the beam to interact only with alternate cavities. The phase change per cavity is then $\pi/2$, giving good mode separation. At the $\pi/2$-mode, losses in the unloaded cavities are extremely small so that the structure is very efficient.

The unloaded cavities can be considered as coupling elements between the accelerating cavities. If the two types of cavities are not identical, then the chain of cavities form a doubly periodic structure.

Nagle (11) has shown that in a lossless doubly periodic structure the π-mode can be obtained for two frequencies, namely the resonant frequencies of the cavities that form the structure.

It is shown here that if losses are introduced in this structure, the edges of the stop band becomes undefined and the π-mode will now occur at one frequency. The dispersion curve is continuous at this frequency which means that, in general, the group velocity is not zero. In fact, the group velocity at the π-mode will be related to the amount of losses and a large mode separation may exist in a lossy structure.

In the dielectrically loaded structure of Walker and West, the dielectric disks may be considered as coupling elements between the accelerating air regions. An equivalent circuit for the disk loaded wave guide has been found and a comparison has been made with the coupled cavity chain.

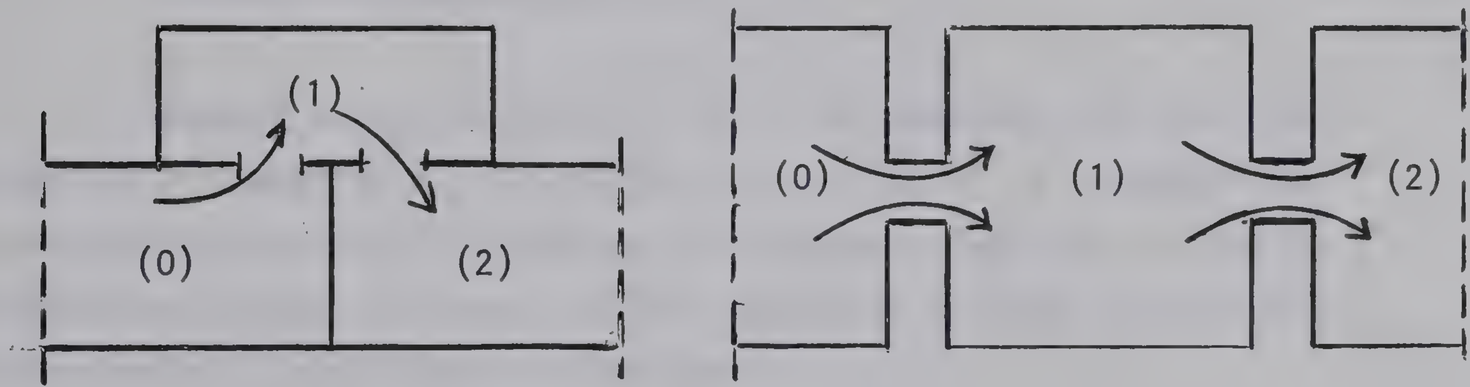
CHAPTER 1

SINGLY PERIODIC STRUCTURE

In this chapter, an equivalent circuit for coupled cavities is assumed, and the transmission matrix for a periodic section is derived. Matrix algebra is used to find the current in each cavity when the structure is terminated.

1.1 SINGLY PERIODIC STRUCTURE

A chain of identical cavities (ie. "same parameters") coupled in a similar manner forms a singly periodic slow wave structure. The cavities could be arranged as in Fig. 1 (a) or (b).



(a) (b)

Fig. 1: Two types of geometrical arrangements are shown for identically coupled cavities:

(a) The cavities are coupled in an off-set manner.

(b) (Linear coupling) Inline coupling.

1.2 EQUIVALENT CIRCUIT

An equivalent circuit can be drawn for the cavities as shown in Fig. 2. This approach is useful so long as it is possible to describe the behavior using separated modes of a single cavity as they develop into bands of the coupled system.

Agreement with measurement is remarkably good as shown by (5), (6), (7), and (8).

The coupling between the cavities is represented by the mutual inductance M .

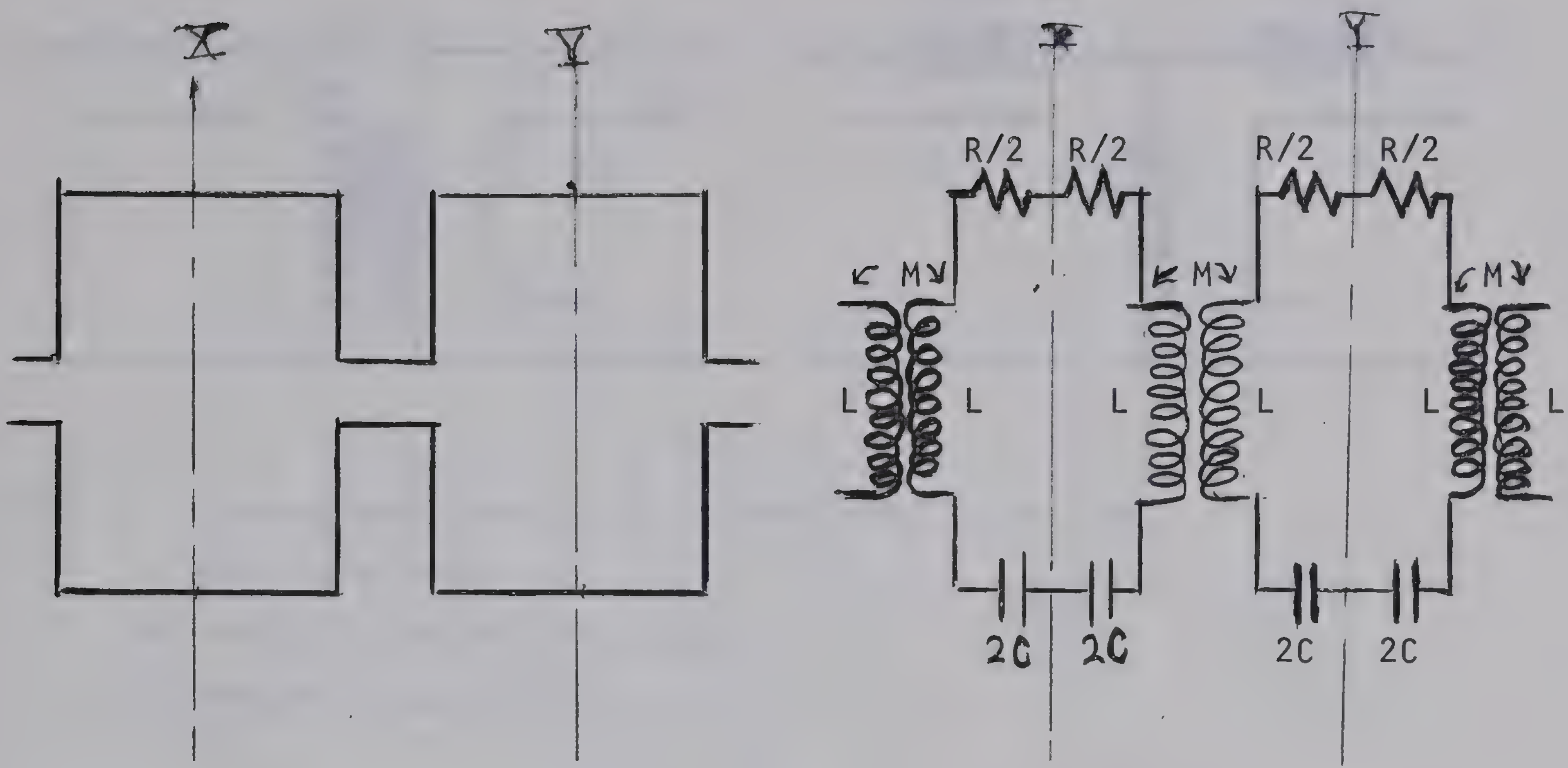


Fig. 2: The equivalent circuit of two coupled cavities with identical parameters.

Depending on the sign of M, the coupling can be either inductive or capacitive. A change in the sign of M changes the relative directions of the current in adjacent cavities. This is illustrated in Fig. 3, where the dot notation is used to determine the equivalent circuit of a transformer:

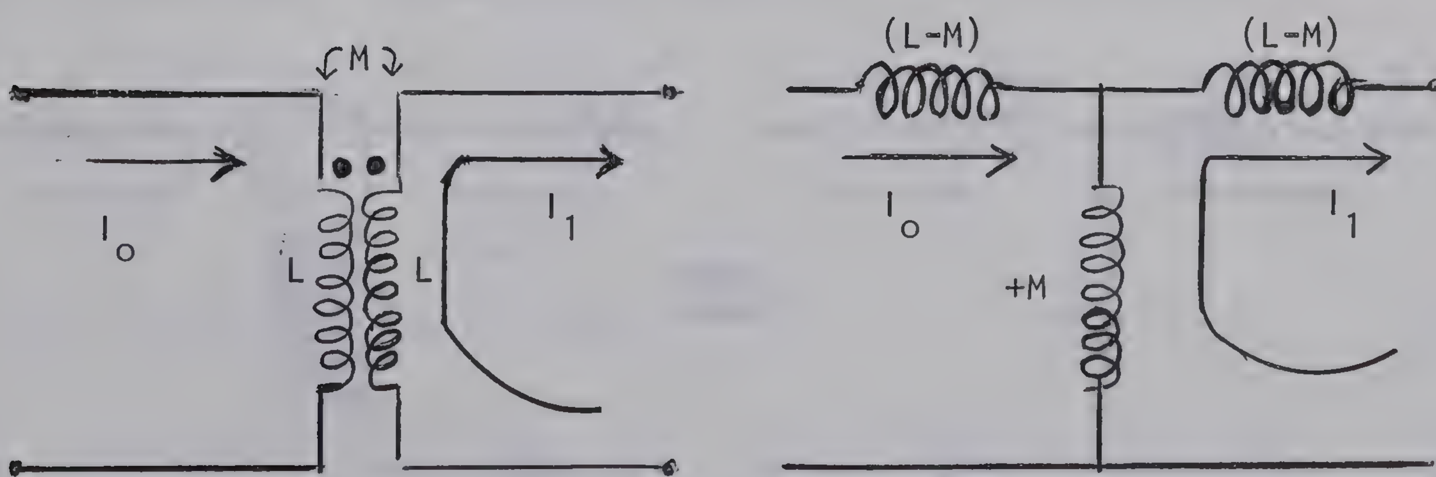


Fig. 3: (a)

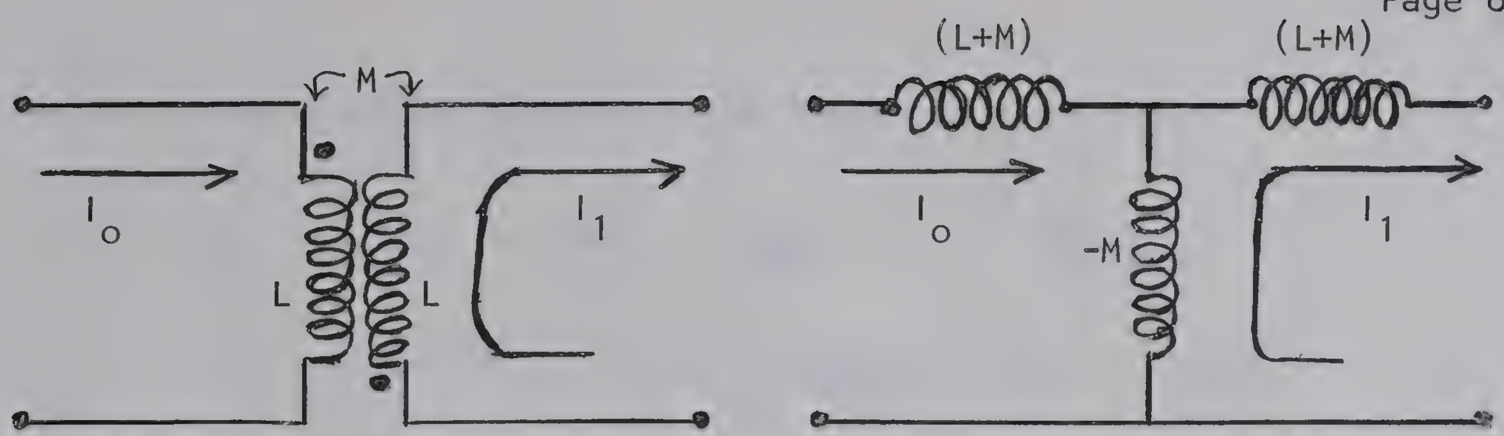


Fig. 3: (b)

Fig. 3: Equivalent curcuit of a transformer using the Dot notation:

(a) Positive mutual inductance.

(b) Negative mutual inductance.

The dispersion curve will be shifted by 180° when M changes sign. Positive sign of M indicates inductive coupling while the negative sign indicates a capacitive one. In the following analysis, capacitive coupling has been assumed.

1.3 TRANSMISSION MATRIX OF A PERIODIC SECTION

A periodic section is shown in Fig. 2 between planes X and Y. It is assumed that the structure has (N) cavities. A particular cavity is denoted by (n). A periodic section therefore consists of one half the cavity (n), and one half of cavity (n+1). Fig. 4 below shows the equivalent circuit of a periodic section.

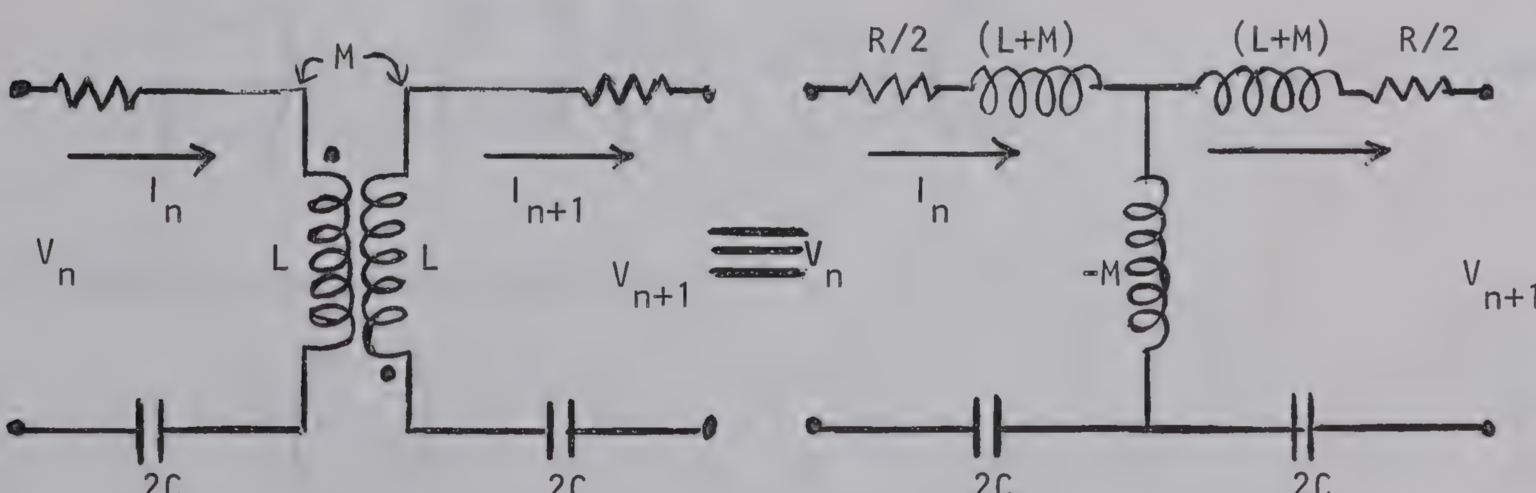


Fig. 4: Equivalent circuit of a periodic section.

From this, using phasor notations, we have:

$$V_n = \left(R/_2 + jWL + \frac{1}{2jWC} \right) I_n + MjWI_{n+1}$$

$$V_{n+1} = -\left(R/_2 + jWL + \frac{1}{2jWC} \right) I_{n+1} - MjWI_n$$

Solving for V_{n+1} and I_{n+1} in terms of V_n and I_n, we get:

$$V_{n+1} = \frac{-(R/_2 + jWL + \frac{1}{2jWC})}{MjW} V_n + \frac{(R/_2 + jWL + \frac{1}{2jWC})^2 - (MjW)^2}{MjW} I_n \qquad (1)$$

$$I_{n+1} = \frac{1}{MjW} V_n - \frac{(R/_2 + jWL + \frac{1}{2jWC})}{MjW} I_n \qquad (2)$$

If we define:

$$A = D = \frac{-(R/_2 + jWL + \frac{1}{2jWC})}{MjW}$$

$$B = \frac{(R/_2 + jWL + \frac{1}{2jWC})^2 - (MjW)^2}{MjW} \qquad (3)$$

$$C = \frac{1}{MjW}$$

The equations (1) and (2) can be written in matrix form giving:

$$\begin{pmatrix} V_{n+1} \\ I_{n+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_n \\ I_n \end{pmatrix} = T \begin{pmatrix} V_n \\ I_n \end{pmatrix} \qquad (4)$$

so that the transmission matrix is:

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (5)$$

Since the equivalent circuit of a periodic section contains only passive elements, the determinent of T must be unity.

$$|T| = AD - BC = 1 \tag{6}$$

From equation (3):

$$AD\text{-}BC = \frac{\left({}^{R}/_{2} + jWL + \frac{1}{2jWC}\right)}{MjW} - \frac{\left({}^{R}/_{2} + jWL + \frac{1}{2jWC}\right)^2}{MjW} + \frac{(MjW)^2}{(MjW)^2} = 1$$

1.4 PHASE SHIFT

The phase shift per section may be found by considering impedances. Because the structure is periodic, the impedance looking into any section has a definite value Z. Thus in Fig. 5, for one section we have:

$$Z_{in} = Z_{out} = Z$$

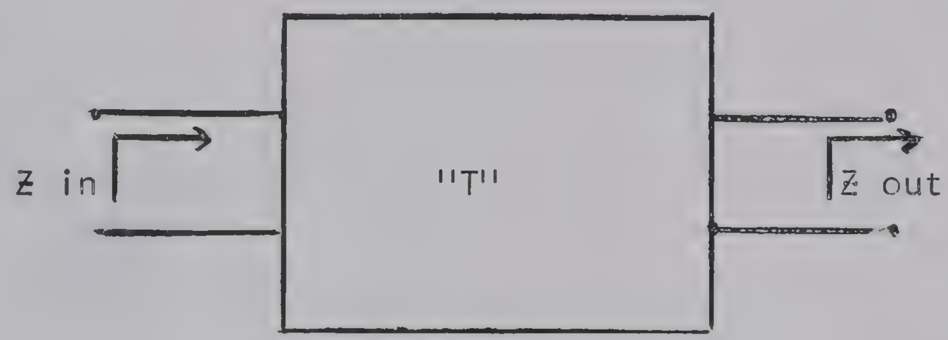


Fig. 5: Schematic diagram for a periodic section.

Therefore, equation (4) becomes:

$$\begin{pmatrix} I_{n+1} Z \\ I_{n+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_n Z \\ I_n \end{pmatrix}$$

so that:

$$I_{n+1} Z = (AZ + B)\, I_n \tag{7}$$

$$I_{n+1} = (CZ + D)\, I_n \tag{8}$$

These are two homogeneous linear equations and a non-trivial solution exists if and only if the determinant is equal to zero.

$$\text{(ie.)} \qquad \begin{vmatrix} Z & -(AZ + B) \\ 1 & -(CZ + D) \end{vmatrix} = 0$$

Expanding and solving for Z we have:

$$Z = \frac{-(D-A)}{2C} \pm \sqrt{\left(\frac{D-A}{2C}\right)^2 + \frac{B}{C}}$$

Since Det T=1 and if $\left(\frac{A+D}{2}\right)^2 \leqslant 1$ (9)

or $\left|\frac{A+D}{2}\right| \leqslant 1$

then:

$$Z = \frac{-(D-A)}{2C} \pm \frac{j}{C}\sqrt{1 - \left(\frac{A+D}{2}\right)^2} \tag{10}$$

This is the characteristic impedance of the structure. Substituting equation (10) in equation (8) gives:

$$I_{n+1} = \left(\frac{A+D}{2} \pm j\sqrt{1 - \left(\frac{A+D}{2}\right)^2}\right) I_n = e^{\pm j\phi} I_n \tag{11}$$

where (+,-) sign indicate forward or backward propagation.
From (11) it is clear that:

$$\phi = \cos^{-1} \frac{(A+D)}{2} \tag{12}$$

which may be real, or complex, depending on A and D. Equation (11) is nothing more than "Floquet's Theorem" for a periodic structure. Equation (12) is the dispersion relation for the structure. For a typical lossless circuit ϕ can be plotted against W to give a dispersion curve as shown in Fig.6.

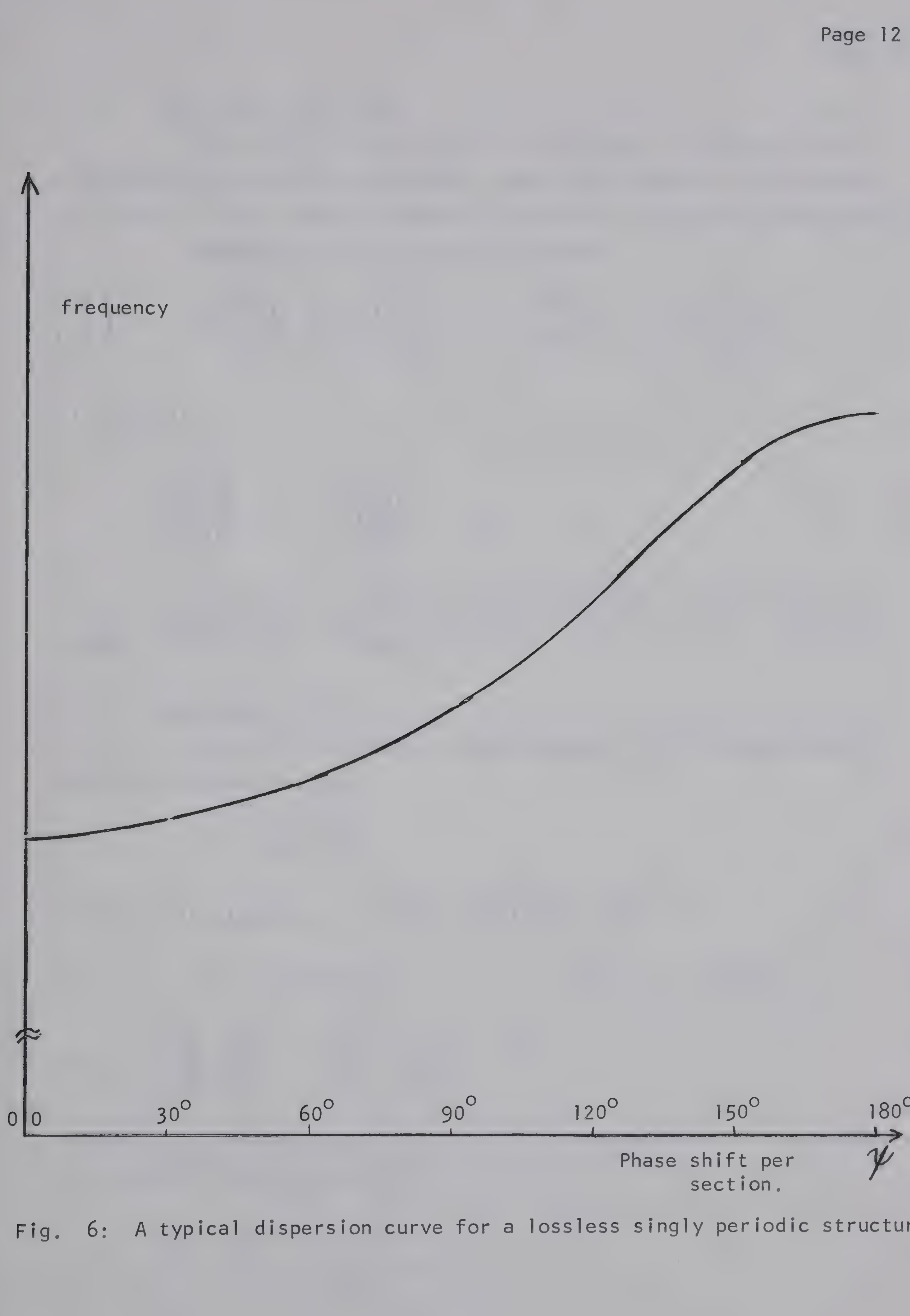


Fig. 6: A typical dispersion curve for a lossless singly periodic structure.

1.5 TERMINATED STRUCTURE

Equation (11) indicates that a forward or a backward wave may exist in a periodic structure. When the structure is terminated, a mixture of both waves is required to satisfy the boundary conditions.

Equation (4) for n=o and n=1 gives:

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = T \begin{pmatrix} V_o \\ I_o \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = T \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = T^2 \begin{pmatrix} V_o \\ I_o \end{pmatrix}$$

In general:

$$\begin{pmatrix} V_n \\ I_n \end{pmatrix} = T^n \begin{pmatrix} V_o \\ I_o \end{pmatrix} \tag{13}$$

Therefore, to relate voltage and current in cavity "n" to those in cavity "o", T^n must be found.

1.6 THE MATRIX T^n

To find T^n, a unitary transformation which diagonalizes T must be found such that:

$$T = ST'S^{-1} \tag{14}$$

where T' is diagonal. If (14) is satisfied, then:

$$T^n = ST'S^{-1}ST'S^{-1} \cdot \cdot \cdot \cdot \cdot \cdot ST'S^{-1} = ST'^nS^{-1} \tag{15}$$

or

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^n = S \begin{pmatrix} \lambda_1^n & o \\ o & \lambda_2^n \end{pmatrix} S^{-1} \tag{16}$$

where $\lambda_{1,2}$ are the eigen values of T, these are found from equating the secular equation determinant to zero:

$$\begin{vmatrix} A - \lambda & B \\ C & D - \lambda \end{vmatrix} = 0 \tag{17}$$

Solving for λ and remembering that DET T =1, we have:

$$\lambda_{1,2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A+D}{2}\right)^2 - 1} = \frac{A+D}{2} \pm j\sqrt{1 - \left(\frac{A+D}{2}\right)^2} = e^{\pm j\phi} \tag{18}$$

where $\left|\frac{A+D}{2}\right| \leqslant 1$ is assumed as before.

and $\phi = \cos^{-1}\left(\frac{A+D}{2}\right)$ which is the same as equation (12).

The Matrix S can be constructed from the eigen vectors of T. Taking $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ as the eigen vector corresponding to λ_1.

(ie.) $\begin{pmatrix} A-\lambda_1 & B \\ C & D-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we have $(A-\lambda_1)x_1 + Bx_2 = 0$.

If we take $(x_2 = 1)$ we find $x_1 = -\frac{B}{A-\lambda_1}$ so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{B}{-A+\lambda_1} \\ 1 \end{pmatrix}$

Similarly, taking $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ as the eigen vector for λ_2 and assuming $(x_4 = 1)$, we get:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{-B}{A-\lambda_2} \\ 1 \end{pmatrix}$$

so that S may be taken as:

$$S = \begin{pmatrix} \frac{-B}{A-\lambda_1} & \frac{-B}{A-\lambda_2} \\ 1 & 1 \end{pmatrix} \tag{19}$$

The inverse of S exists if $|S| \neq 0$ (ie.) $\frac{B}{A-\lambda_2} - \frac{B}{A-\lambda_1} \neq 0$.

Inverting, we have:

$$S^{-1} = \frac{1}{\frac{B}{A-\lambda_2} - \frac{B}{A-\lambda_1}} \begin{pmatrix} 1 & \frac{B}{A-\lambda_2} \\ -1 & \frac{-B}{A-\lambda_1} \end{pmatrix} \tag{20}$$

Substituting in equation (16), we get:

$$T^n = \begin{pmatrix} \dfrac{A(\lambda_2^n - \lambda_1^n) - \lambda_1\lambda_2(\lambda_2^{n-1} - \lambda_1^{n-1})}{\lambda_2 - \lambda_1} & \dfrac{B(\lambda_2^n - \lambda_1^n)}{\lambda_2 - \lambda_1} \\ \\ -\dfrac{(A-\lambda_1)(A-\lambda_2)(\lambda_2^n - \lambda_1^n)}{B(\lambda_2 - \lambda_1)} & \dfrac{(\lambda_2^{n+1} - \lambda_1^{n+1}) - A(\lambda_2^n - \lambda_1^n)}{\lambda_2 - \lambda_1} \end{pmatrix} \tag{21}$$

Using the identities:

$$\lambda_1 + \lambda_2 = A + D$$

$$\lambda_1\lambda_2 = AD - BC = 1$$

$$\frac{(\lambda_2^{n+1} - \lambda_1^{n+1}) - A(\lambda_2^n - \lambda_1^n)}{\lambda_2 - \lambda_1} = \frac{(\lambda_2^n - \lambda_1^n)(\lambda_1 + \lambda_2 - A)}{\lambda_2 - \lambda_1} - \frac{\lambda_1\lambda_2(\lambda_2^{n-1} - \lambda_1^{n-1})}{\lambda_2 - \lambda_1}$$

and $\dfrac{-(A-\lambda_1)(A-\lambda_2)}{B} = C, \qquad \lambda_{1,2} = e^{\pm j\phi}$ (22)

equation (21) becomes:

$$T^n = \begin{pmatrix} A\dfrac{\sin n\phi - \sin(n-1)\phi}{\sin\phi} & B\dfrac{\sin n\phi}{\sin\phi} \\ \\ C\dfrac{\sin n\phi}{\sin\phi} & D\dfrac{\sin n\phi - \sin(n-1)\phi}{\sin\phi} \end{pmatrix} \tag{23}$$

If we define: $A_n = A\dfrac{\sin n\phi - \sin(n-1)\phi}{\sin\phi}$ $\qquad C_n = C\dfrac{\sin n\phi}{\sin\phi}$

$$B_n = B\frac{\sin n\phi}{\sin\phi} \qquad D_n = D\frac{\sin n\phi - \sin(n-1)\phi}{\sin\phi}$$

(24)

Then: $$T^n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \qquad (25)$$

1.7 THE CURRENTS IN THE CAVITIES:

A structure consisting of N cavities terminated in an impedance Z_N is shown in Fig. 7:

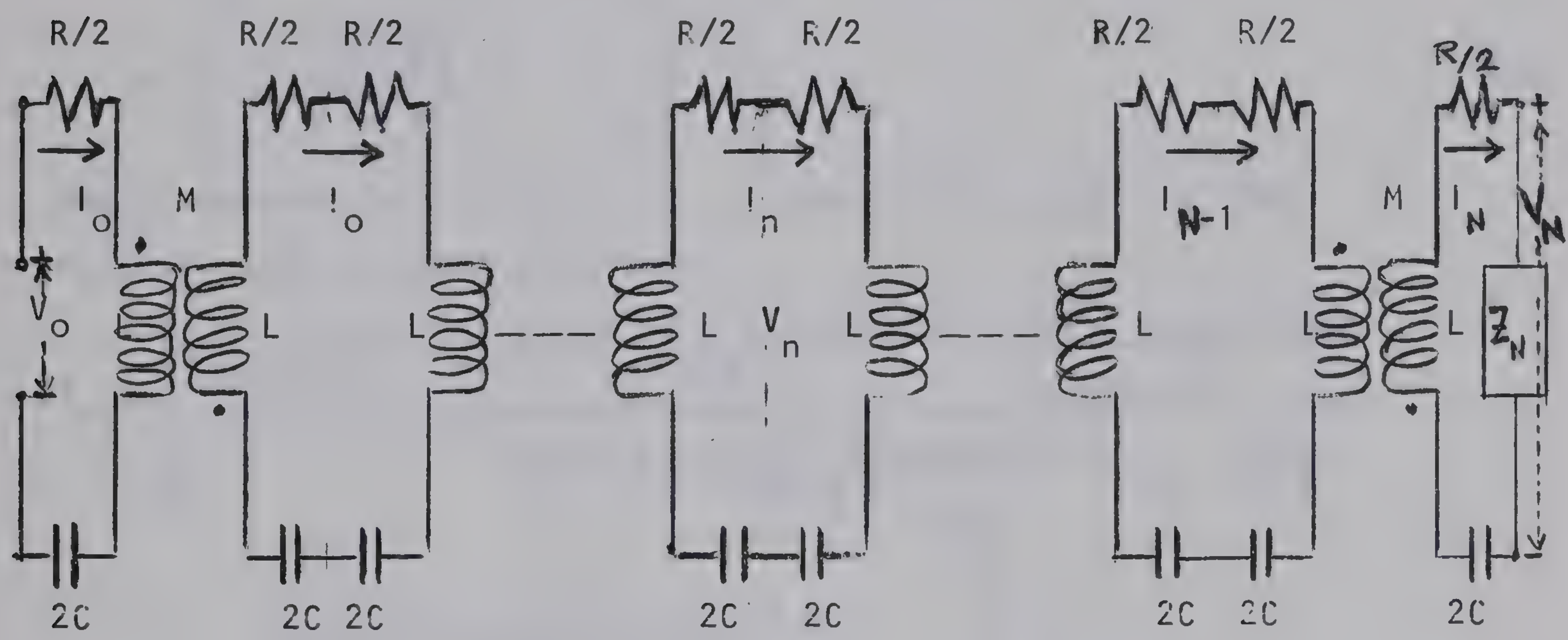


Fig. 7: Equivalent circuit of (N) cavities with termination Z_N.

At the termination: $$V_N = Z_N I_N. \qquad (26)$$

Using equation (13) with n=N, we have:

$$\begin{pmatrix} I_N Z_N \\ I_N \end{pmatrix} = \begin{pmatrix} A_N & B_N \\ C_N & D_N \end{pmatrix} \begin{pmatrix} V_o \\ I_o \end{pmatrix}$$

Solving for V_o in terms of I_o we find:

$$V_o = \left[\frac{Z_N D_N - B_N}{A_N - Z_N C_N} \right] I_o \qquad (27)$$

Substitute equation (27) in (13) gives:

$$\begin{pmatrix} V_n \\ I_n \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \begin{pmatrix} \left[\dfrac{Z_N D_N - B_N}{A_N - Z_N C_N}\right] I_o \\ I_o \end{pmatrix} \tag{28}$$

This equation gives V_n and I_n in terms of I_o. Therefore I_n is:

$$I_n = \left[C_n \left(\frac{Z_N D_N - B_N}{A_N - Z_N C_N}\right) + D_n \right] I_o \tag{29}$$

For a symmetrical structure A=D so that, using equation (24), equation (29) becomes explicitly:

$$I_n = \left[\frac{C\, Z_N \dfrac{\sin n\phi}{\sin\phi}\, \dfrac{\cos\phi \sin N\phi - \sin(N-1)\phi}{\sin\phi} + \dfrac{\sin^2\phi \sin n\phi \sin N\phi}{\sin^2\phi}}{\dfrac{\cos\phi \sin N\phi - \sin(N-1)\phi}{\sin\phi} - Z_N C \dfrac{\sin N\phi}{\sin\phi}} + \frac{\cos\phi \sin n\phi - \sin(n-1)\phi}{\sin\phi} \right] I_o$$

where $A = D = \cos\phi$ from (12) (3).

Simplifying we have:

$$I_n = \left[\frac{Z_N C \dfrac{\sin n\phi}{\sin\phi} \cos N\phi + \sin n\phi \sin N\phi}{\cos N\phi - Z_N C \dfrac{\sin N\phi}{\sin\phi}} + \cos n\phi \right] I_o \tag{30}$$

There are three special cases that may be considered:

(a) When the terminating impedance Z_N is made equal to the characteristic impedance of the structure, then equation (30) will reduce to a general case, of which equation (11) was a special case.
The characteristic impedance is given by equation (10).
But since **A** = D, equation (10) reduces to:

$$\bar{Z} = \frac{\pm j}{c} \sqrt{1 - \cos^2 \phi} = \frac{\pm j}{c} \sin \phi \tag{31}$$

Substitution in (30) gives:

$$I_n = \left(\frac{(\pm) j \sin n \phi \cos N \phi + \sin n \phi \sin N \phi}{\cos N \phi - (\pm) j \sin N \phi} + \cos n \phi \right) I_o$$

$$I_n = \left(\cos n \phi \pm j \sin n \phi \right) I_o = I_o e^{\pm j n \phi} \tag{32}$$

which is the statement of Floquet's theorem in an infinitely long periodic structure.

(b) When the terminating impedance is infinite, $\bar{Z}_N = \infty$, then:

$$I_n = \left(\frac{\sin n \phi \cos N \phi}{- \sin N \phi} + \cos N \phi \right) I_o = \frac{\sin (N-n) \phi}{\sin N \phi} I_o \tag{33}$$

(c) If $\bar{Z}_N = 0$, then:

$$I_n = \frac{\sin n \phi \sin N \phi + \cos n \phi \cos N \phi}{\cos N \phi} I_o = \frac{\cos (N-n) \phi}{\cos N \phi} I_o \tag{34}$$

Equation (34) is of special importance as shown in the next chapter.

CHAPTER II

DISPERSION CURVE, GROUP VELOCITY, AND CURRENTS IN A SINGLY PERIODIC STRUCTURE

In this chapter the currents in the cavities are determined for the case where the termination is a short circuit. The dispersion curve and group velocity are also calculated for a specific structure using equations (34) and (12). Two cases are considered:

(a) An ideal lossless structure.

and (b) A lossy structure.

2.1 CURRENTS IN THE CAVITIES

The currents in the cavities, subject to a short circuit termination, are given by equation (34). In general, the phase shift per section is a complex quantity. If $e^{-j\phi}$ is taken to represent the phase shift per section of a wave travelling to the right, then for a lossy structure ϕ should be of the form

$$\phi = \psi - j\alpha \tag{35}$$

where ψ, α are real quantities. ψ is the real phase shift per section and α is a positive quantity defined such that $e^{-\alpha}$ is the amplitude attenuation of the wave per section. Since ϕ is complex $\cos\phi$ is given by the following identity:

$$\cos\phi = \cos(\psi - j\alpha) = \cos\psi\cosh\alpha + j\sin\psi\sinh\alpha \tag{36}$$

Substituting (36) in (34) gives:

$$I_n = \left(\frac{\cos(N-n)\psi\cosh(N-n)\alpha + j\sin(N-n)\psi\sinh(N-n)\alpha}{\cos N\psi\cosh N\alpha + j\sin N\psi\sinh N\alpha}\right) I_o \tag{37}$$

2.2 CURRENTS AT THE π -MODE IN A LOSSLESS STRUCTURE.

In a lossless structure, $\alpha = o$, so that equation (37) reduces to:

$$I_n = \frac{\cos (N-n)\psi}{\cos N\psi} I_o \tag{38}$$

If the chain of cavities is short circuited at both ends it will resonate when the total ~~pulse~~ phase shift is a multiple of π.

$$\text{(ie)} \quad N\psi = q\pi \quad ; \quad q = o, 1, 2 \ldots . N \tag{39}$$

Equation (39) shows that the phase shift is discrete and is given by $\psi = q\pi/N$. There are (N+1) values for the phase shift corresponding to (q = o, 1, 2. . . . N). Fig. 8 below shows how the resonant frequencies can be found from the dispersion curve.

Here N=4 has been considered. The difference between adjacent resonant frequencies is known as the mode separation and it is related to the slope of the dispersion curve. It can be seen from Fig. 8 that the mode separation is large at $\psi = \pi/2$ and smallest for $\psi = \pi$.

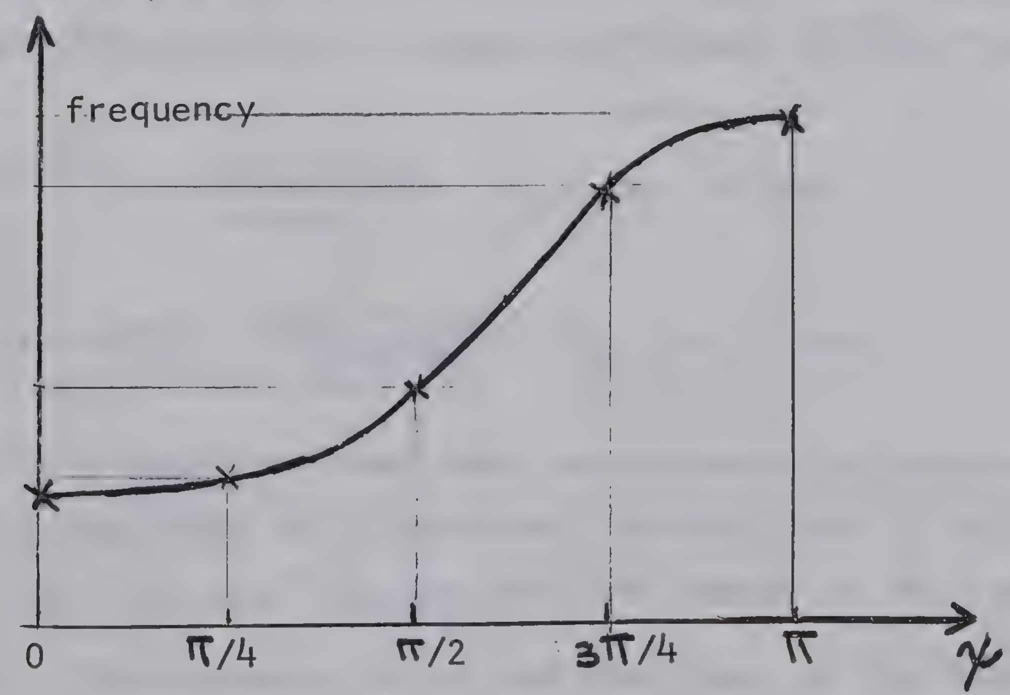


Fig. 8: Resonant frequencies for a four-section structure.

If ψ is to have the value $\frac{\pi}{2}$, then by equation (39) N must be an even integer.

Putting ψ equal to $\pi/2$ in (38) gives:

$$I_n = \cos n\frac{\pi}{2} \; I_o \tag{40}$$

Equation (40) above indicates that the currents are zero in the odd cavities and are equal in the magnitude to the applied current I_o in the even cavities. Hence, only the even cavities are excited and the odd cavities may be considered as lossless coupling elements.

In a periodic structure whose cavities are arranged as in Fig. 1 (a), the even cavities may be used as a particle accelerator. The particle beam will experience π -mode operation, but the mode separation will be large because the total structure is operating at the $\pi/2$ mode.

2.3 CURRENTS AT THE π/2-MODE IN A LOSSY STRUCTURE.

When losses are present in the structure at the $\pi/2$ mode, the currents in the odd cavities are no longer zero. Equation (37) with the conditions that N is even and ψ equal to $\pi/2$, reduces to:

$$I_n = \begin{cases} \cos n\pi/2 \quad \dfrac{\cosh (N-n)\alpha}{\cosh N\alpha} \quad I_o; & \text{for } n \text{ even} \\ \\ -j \sin n\pi/2 \quad \dfrac{\sinh (N-n)\alpha}{\cosh N\alpha} \quad I_o; & \text{for } n \text{ odd} \end{cases} \tag{41}$$

This equation shows that the currents in the even cavities decrease in amplitude as n increases, varying from I_o for n = o to $\frac{I_o}{\cosh N\alpha}$ for n = N. For small values of α the change in the current is negligible. The currents in the odd cavities, on the other hand, are proportional to $(N-n)\alpha$ for small values of α, and hence, decrease linearly to zero as the termination is approached.

2.4 DISPERSION RELATION AND THE GROUP VELOCITY.

Equation (12) is the dispersion relation for any periodic structure. For the structure under consideration

$A = D = -\left(\dfrac{R/2 + jWL + \frac{1}{jWC}}{MjW}\right)$ from equation (3). Using this in

equation (12) gives the dispersion relation explicitly in terms of the frequency:

$$\cos \phi = A = D = -\frac{(R/2 + jWL + \frac{1}{2jWC})}{MjW} \tag{42}$$

We may define: $W_o^2 = \dfrac{1}{2LC}$ $\quad Q = \dfrac{2LW_o}{R}$ (43)

Here, W_o is the resonant frequency of one of the cavities since the total inductance of each cavity is 2L. The mutual inductance "M" which represents the coupling mechanism may be written in terms of the inductances and a coupling factor "k". It is known from transformer theory that:

$$M = k\sqrt{L_1 L_2} = kL \tag{44}$$

Substituting equations (43), (44), in (42) and separating the real and imaginary parts gives:

$$\cos \phi = \frac{1}{k}\left[\left(\frac{W_o}{W}\right)^2 - 1\right] + j\,\frac{1}{kQ}\,\frac{W_o}{W} \tag{45}$$

The above equation shows that ϕ must be a complex quantity, in agreement with equation (35).

Equating real and imaginary parts of equations (45), and (36) gives:

$$\cos \psi \cosh \alpha = \frac{1}{k}\left[\left(\frac{W_o}{W}\right)^2 - 1\right] \tag{46}$$

and

$$\sin \psi \sinh \alpha = \frac{1}{kQ}\,\frac{W_o}{W} \tag{47}$$

These two equations determine the dispersion curve completely for a singly periodic structure. Only the parameters W_o, Q, and k are necessary to plot the dispersion curve.

2.5 DISPERSION CURVE AND GROUP VELOCITY FOR A LOSSLESS STRUCTURE.

If the singly periodic structure is lossless (ie) "$Q=\infty$" then the equations (46) and (47) reduce to one equation, namely:

$$\cos\psi = \frac{1}{k}\left(\left(\frac{W_o}{W}\right)^2 - 1\right) \tag{48}$$

This is the dispersion relation for a lossless singly periodic structure. Solving for $\frac{W}{W_o}$ in terms of ψ one finds:

$$\frac{W}{W_o} = \left[1 + k\cos\psi\right]^{-\frac{1}{2}} \tag{49}$$

Hence, the frequency is a periodic function of ψ with a period equal to "π". The dispersion curve is, therefore, as shown in Fig. 9 with the pass band extending from $W = \frac{W_o}{\sqrt{1+k}}$ to $W = \frac{W_o}{\sqrt{1-k}}$

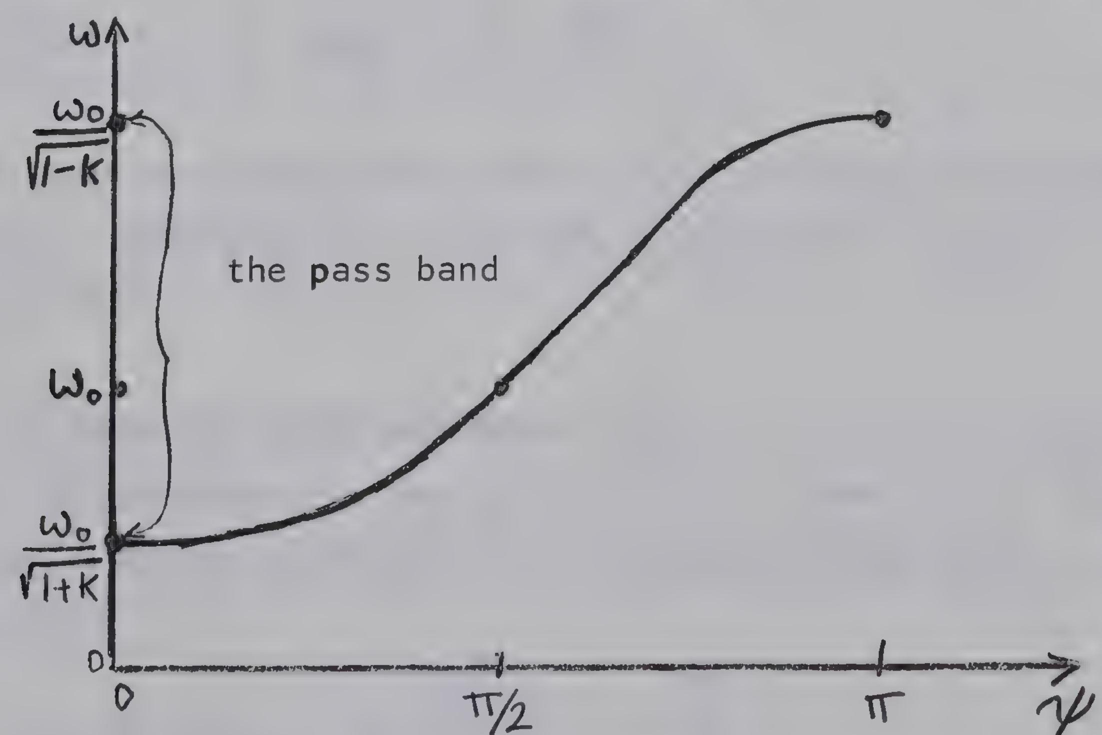


Fig. 9: The dispersion curve of a lossless singly periodic structure.

2.6 MODE SEPARATION.

We have seen in Section 2.2 that the mode separation is related to the slope of the dispersion curve. The mode separation is approximately proportional to $dw/d\psi$. It can be noted here that the group velocity, (ie) the velocity at which information travels along the structure, is also proportional to this quantity.

Differentiating equation (49) gives:

$$\frac{dW}{d\psi} = \frac{W_o}{2} k \sin\psi \left[1 + \cos\psi\right]^{-3/2} = \frac{kW^3}{2W_o^2} \sin\psi \tag{50}$$

for $\psi = \pi/2$, $W = W_o$ and:

$$\frac{dW}{d\psi} = \frac{k}{2} W_o \tag{50-a}$$

Equation (50-a) shows that the group velocity at $\psi = \pi/2$ is proportional to the resonant frequency of the cavity and the coupling constant. The percentage mode separation at the $\pi/2$ mode is given by

$$\frac{\delta W}{W_o} = \frac{k}{2} d\psi = \frac{k}{2} \frac{\pi}{N}$$

Thus, for good mode separation, the value of k cannot be too small when N is large. When $\psi \neq \pi/2$ the mode separation is reduced as indicated by equation (50).

2.7 DISPERSION CURVE AND GROUP VELOCITY IN A LOSSY STRUCTURE.

The solution for the phase shift in terms of the frequency in a lossy structure is obtained by eliminating (α) between equations (46) and (47):

$$\cos\psi \cosh\alpha = \frac{1}{k}\left[\left(\frac{W_o}{W}\right)^2 - 1\right]$$

$$\sin\psi \sinh\alpha = \frac{1}{kQ} \frac{W_o}{W}$$

Substituting for $\cosh\alpha$ from the second equation into the first gives:

$$\left(\frac{W_o}{W}\right)^4 - \left(2 + \frac{\cot^2\psi}{Q}\right)\left(\frac{W_o}{W}\right)^2 - k^2\cos^2\psi + 1 = 0$$

Solving for $\left(\frac{W_o}{W}\right)^2$ one obtains:

$$\left(\frac{W_o}{W}\right)^2 = \begin{cases} 1 + \dfrac{\cot^2\psi}{2Q^2} + \sqrt{\left(1 + \dfrac{\cot^2\psi}{2Q^2}\right)^2 + k^2\cos^2\psi - 1}\ ; \text{for } 0 \leq \psi \leq \pi/2 \\ 1 + \dfrac{\cot^2\psi}{2Q^2} - \sqrt{\left(1 + \dfrac{\cot^2\psi}{2Q^2}\right)^2 + k^2\cos^2\psi - 1}\ ; \text{for } \pi/2 \leq \psi \leq \pi \end{cases} \tag{51}$$

In the limit as $Q \rightarrow \infty$ equations (51) reduces to equation (49). For small losses the dispersion curve is almost the same as that of a lossless structure except at $(\psi = 0, \pi)$. As $\psi \rightarrow 0$, $W \rightarrow 0$, and as $\psi \rightarrow \pi$, $W \rightarrow \infty$ as shown in Fig. (10) and (11). The group velocity can be found by differentiating equations (46) and (47) and eliminating $\frac{d\alpha}{dW}$ between the resulting equations. Thus:

$$-\frac{d\psi}{dW}\sin\psi\cosh\alpha + \cos\psi\sinh\alpha\frac{d\alpha}{dW} = -2\frac{W_o^2}{kW^3}$$

and $$\cos\psi\sinh\alpha\frac{d\psi}{dW} + \sin\psi\cosh\alpha\frac{d\alpha}{dW} = -\frac{W_o}{kQW^2}$$

Eliminating $\frac{d\alpha}{dW}$ we have:

$$\frac{dW}{d\psi} = kQW^3\left(\frac{\sin^2\psi + \sinh^2\alpha}{2QW_0^2\sin\psi\cosh\alpha - WW_o\cos\psi\sinh\alpha}\right) \tag{52}$$

Comparison of equations (50) and (52) indicates that the group velocity in a lossy structure is always greater than the group velocity in a lossless structure. This means that the dispersion curve in the lossy case is always steeper than the one with no lossess. At $\psi = \pi$, this difference is very large since the tangent of the lossy curve is vertical while that of the lossless curve is horizontal. At $\psi = \pi/2$, equation (52) gives:

$$\left.\frac{dW}{d\psi}\right|_{\psi = \pi/2} = \frac{k}{2} W_o \sqrt{1 + \frac{1}{kQ^2}} \tag{53}$$

Since from (51) we have $W = W_o$, $\cosh\alpha$ has been substituted from equation (47).

Comparison of (53) with (50-a) shows that the group velocity has increased by the amount $\sqrt{1 + \frac{1}{k^2Q^2}}$ when losses are introduced into the structure. A tabulation of $\left(\frac{W}{W_o}\right)$ versus the phase shift is shown in the following table for different values of the coupling constant and different Q . The dispersion curves are plotted in Fig. (10) and (11). In the next chapter these ideas are applied to the doubly periodic structure.

TABLE 4

VALUES OF $\frac{W}{W_o}$ FOR DIFFERENT VALUE OF ψ WHEN k AND Q ARE GIVEN VARIOUS VALUES.

		ψ								
		0°	15°	30°	60°	90°	120°	150°	165°	180°
$\frac{W}{W_o}$	k=.5 Q=∞	0.815	—	0.835	0.895	1	1.15	1.335	—	1.41
	k=.5 Q=10	0	0.77	0.825	0.892	1	1.116	1.345	1.49	∞
	k=.5 Q=5	0	0.67	0.785	0.88	1	1.17	1.4	1.71	∞
	k=.2 Q=∞	0.91	—	0.923	0.95	1	1.05	1.1	—	1.118
	k=0.2 Q=10	0	0.815	0.894	0.95	1	1.06	1.14	1.24	∞
	k=0.2 Q=5	0	0.69	0.89	0.93	1	1.08	1.2	1.48	∞

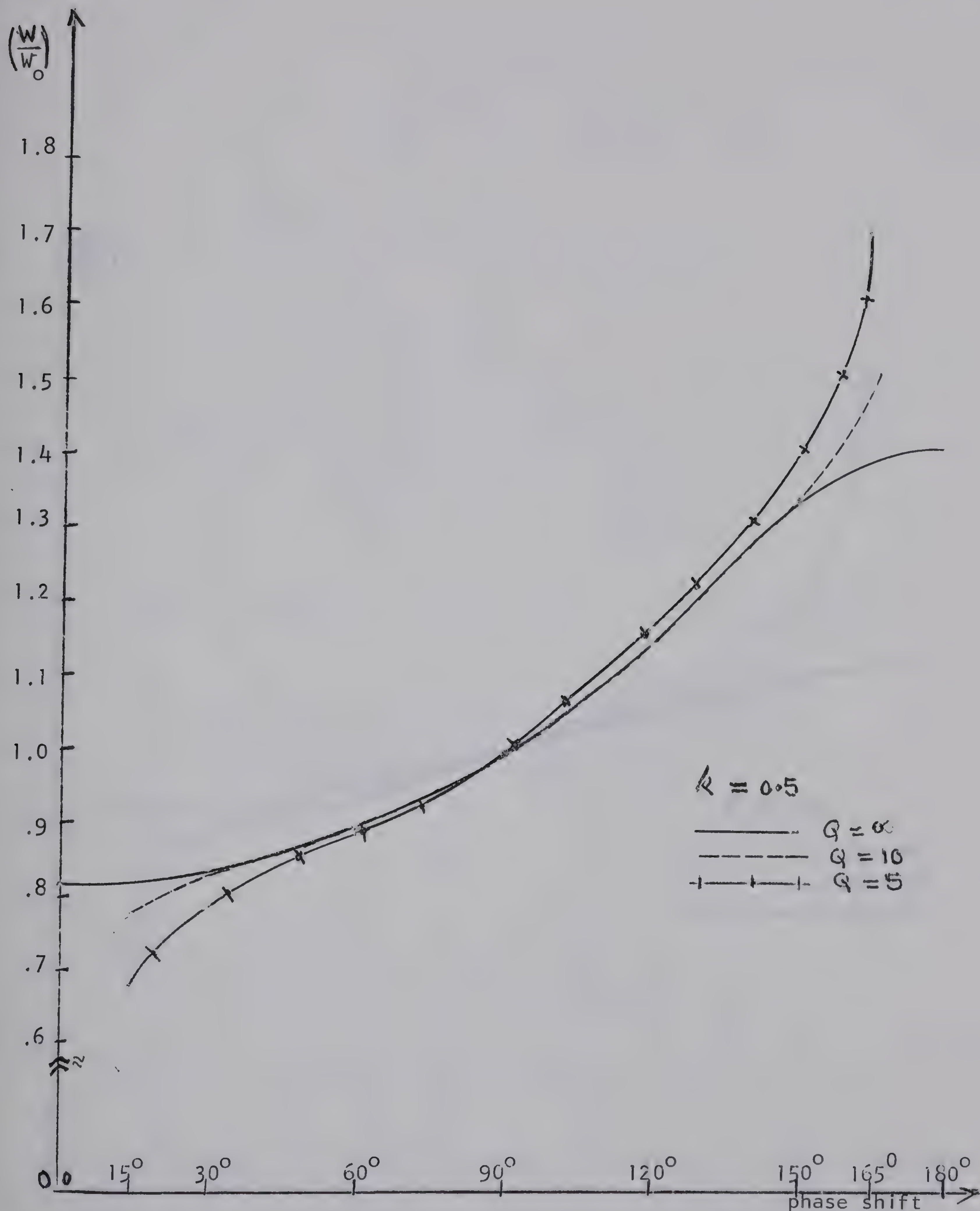


Fig. 10: The dispersion curve with and without loses for a singly periodic structure.

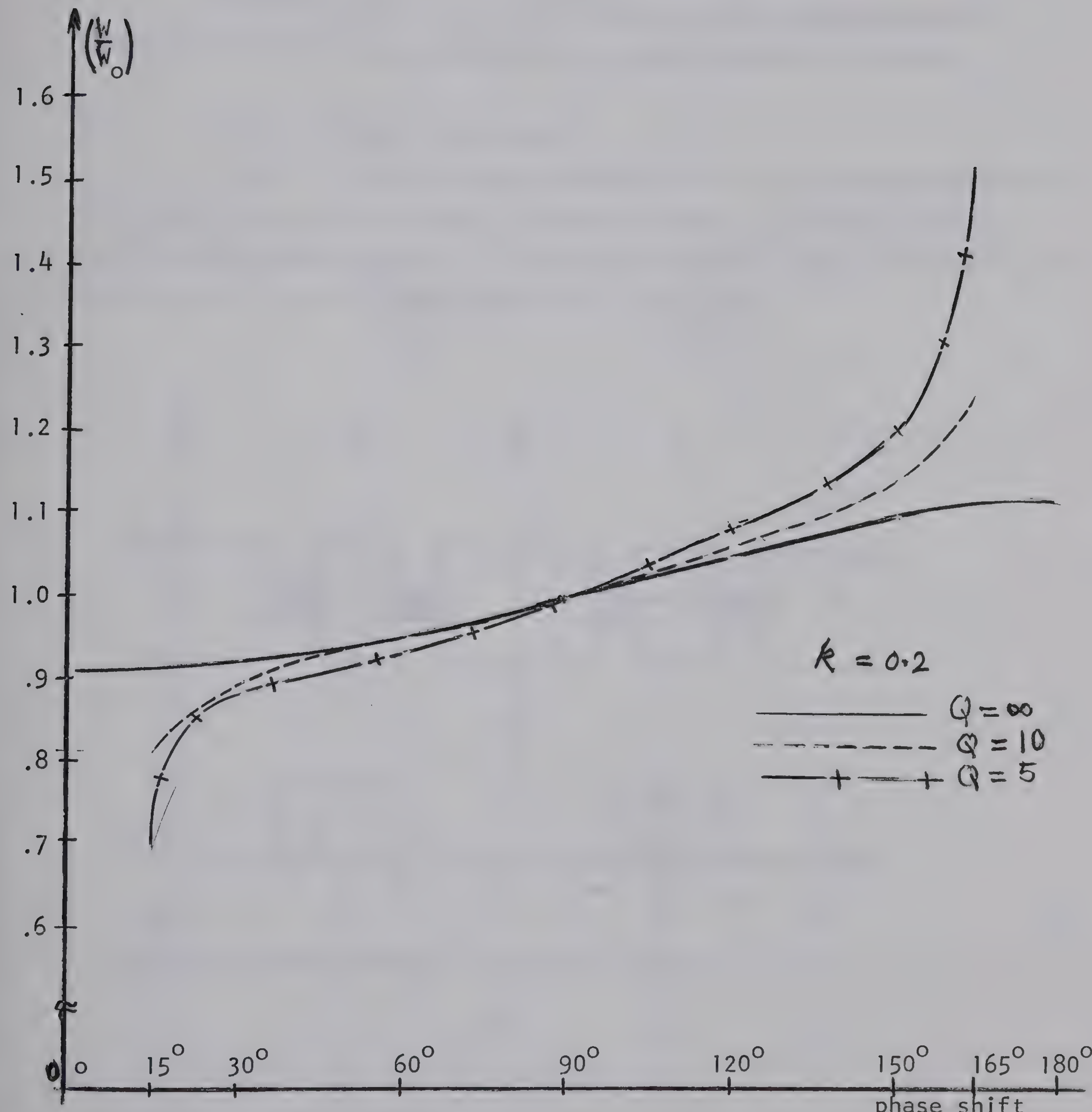


Fig. 11: The dispersion curve with and without losses for a singly periodic structure.

CHAPTER III

DOUBLY PERIODIC STRUCTURE

The theory of a singly periodic structure consisting of identical cavities can be extended to a doubly periodic structure .

3.1 DOUBLY PERIODIC STRUCTURE.

A chain of two different types of cavities arranged alternately and coupled in a similar manner as shown in Fig. 12, forms a doubly periodic slow wave structure. The cavities may be placed "in-line" as in Fig 12-a, or in an "off-set" manner as in Fig. 12-b.

Fig. 12 A doubly periodic structure is shown.
(a) in line coupling.
(b) offset coupling.

3.2 EQUIVALENT CIRCUIT (of a periodic section)

The equivalent circuit of a doubly periodic structure is the same as that of a singly periodic structure except that the odd and even cavities have different parameters. A symmetric periodic section is taken between planes X and Y as shown in Fig. 13.

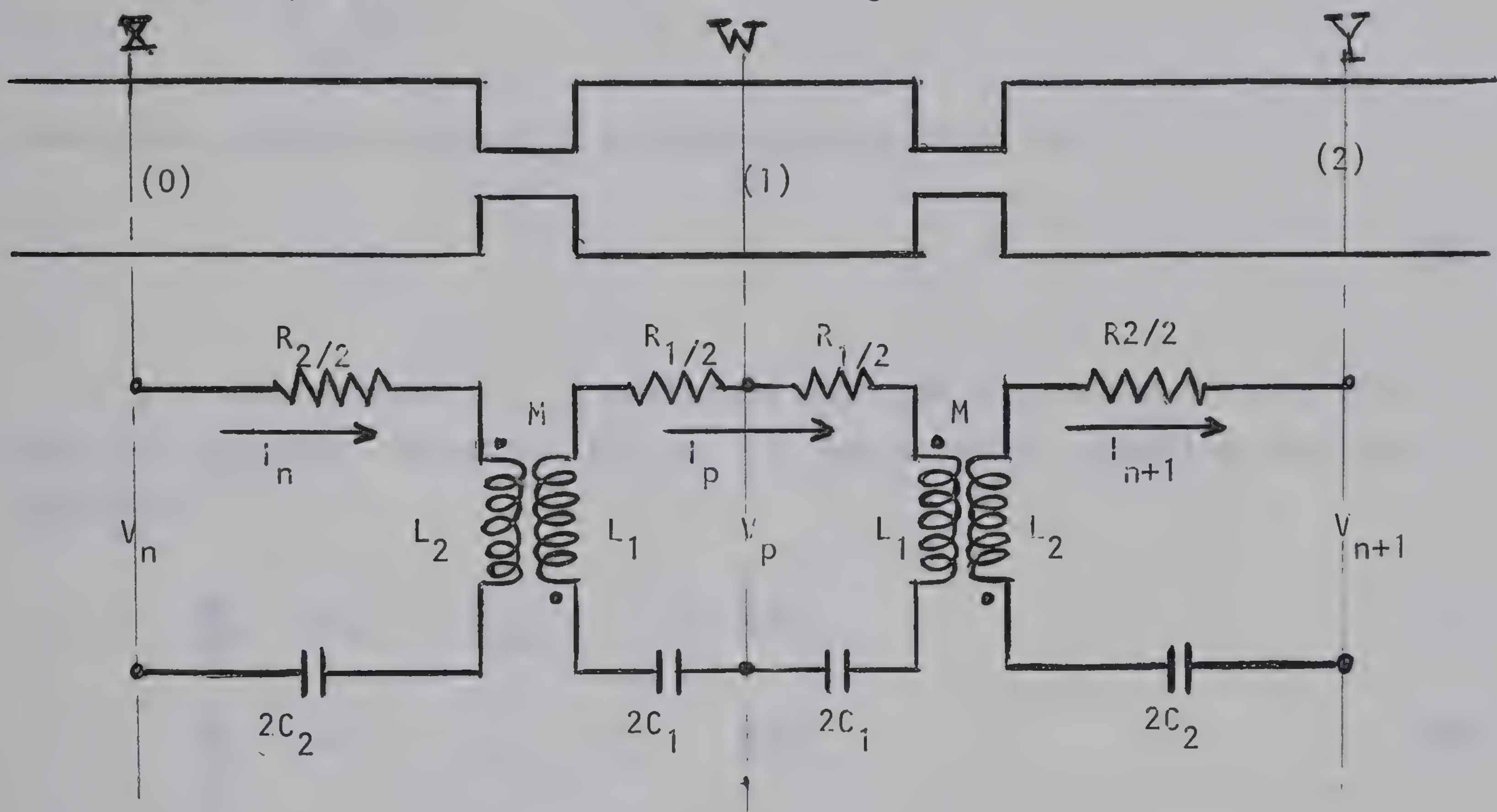


Fig. 13: An equivalent circuit of a periodic section containing two cavities.

3.3 TRANSMISSION MATRIX.

The transmission matrix which relates the input to the output of a periodic section is obtained as follows:

Referring to Fig. 13, we have as in chapter (1):

$$\begin{pmatrix} V_p \\ I_p \end{pmatrix} = T_1 \begin{pmatrix} V_n \\ I_n \end{pmatrix} \tag{54}$$

and
$$\begin{pmatrix} V_{n+1} \\ I_{n+1} \end{pmatrix} = T_2 \begin{pmatrix} V_p \\ I_p \end{pmatrix} = T_2 T_1 \begin{pmatrix} V_n \\ I_n \end{pmatrix} \qquad (55)$$
$$= T \begin{pmatrix} V_n \\ I_n \end{pmatrix}$$

thus, the transmission matrix of a periodic section is:

$$T = T_2 T_1 \qquad (56)$$

The matrices T_1, T_2 are found from the equivalent circuit of a periodic section. Referring to Fig. 13, and assuming capactive coupling, one gets:

$$V_n = \frac{R_1}{2} + jWL_1 + \frac{1}{2jWC_1} \quad I_n + MjWI_p \qquad (57)$$

$$V_p = \frac{R_2}{2} + jWL_2 + \frac{1}{2jWC_2} \quad I_p - MjWI_n \qquad (58)$$

If we let

$$\frac{R_1}{2} + jWL_1 + \frac{1}{2jWC_1} = a_1$$
$$\frac{R_2}{2} + jWL_2 + \frac{1}{2jWC_2} = a_2 \qquad (59)$$
$$MjW = b$$

Equations (57) and (58) can be written simply as:

$$V_n = a_2 I_n + bI_p \qquad (60)$$

$$V_p = -a_1 I_p - bI_n \qquad (61)$$

Solving for V_p and I_p in terms of V_n and I_n we find:

$$\begin{pmatrix} V_p \\ I_p \end{pmatrix} = \begin{pmatrix} \frac{-a_1}{b_1} & \frac{a_1 a_2 - b^2}{b} \\ \frac{1}{b} & \frac{-a_2}{b} \end{pmatrix} \begin{pmatrix} V_n \\ I_n \end{pmatrix} \tag{62}$$

Comparing this with equation (54) gives:

$$T_1 = \begin{pmatrix} -\frac{a_1}{b_1} & \frac{a_1 a_2 - b^2}{b} \\ \frac{1}{b} & \frac{-a_2}{b} \end{pmatrix} \tag{63}$$

Similarly:

$$V_p = a_1 I_p + b\, I_{n+1} \tag{64}$$

$$V_{n+1} = a_2 I_{n+1} - b I_p \tag{65}$$

and following the same steps as in the derivation of (63), one obtains for T_2:

$$T_2 = \begin{pmatrix} \frac{-a_2}{b} & \frac{a_1 a_2 - b^2}{b} \\ \frac{1}{b} & \frac{-a}{b} \end{pmatrix} \tag{64}$$

therefore:

$$T = T_2 T_1 = \begin{pmatrix} \frac{-a_2}{b} & \frac{a_1 a_2 - b^2}{b} \\ \frac{1}{b} & \frac{-a_1}{b} \end{pmatrix} \begin{pmatrix} \frac{-a_1}{b} & \frac{a_1 a_2 - b^2}{b} \\ \frac{1}{b} & \frac{-a_2}{b} \end{pmatrix}$$

Multiplying the matrices we have:

$$T = \begin{pmatrix} \dfrac{2a_1a_2-b^2}{b^2} & -2\left(\dfrac{a_2{}^2a_1-a_2b^2}{b^2}\right) \\ \\ \dfrac{-2a_1}{b^2} & \dfrac{2a_1a_2-b^2}{b^2} \end{pmatrix} \tag{65}$$

Unlike T_1, or T_2, T is a ~~symmetric~~ matrix with $T_{11}=T_{22}$. This is to be expected, since a symmetric periodic section has been chosen. With this choice, therefore, the structure now resembles a singly periodic structure and the theory developed in Chapters (1) and Chapters (2) may be applied.

3.4 CURRENTS IN THE CAVITIES.

According to the labelling used in Fig 13 "n" labels even cavities and "p" labels odd cavities. Thus with a short circuit termination, the currents in the even cavities are given by equation (34):

$$I_n = \frac{\cos (N-n)\, \phi}{\cos N\, \phi} \quad I_o \tag{66}$$

Where ϕ is the phase shift per section which is given by equation (12) using:

$$A = D = \frac{2a_1a_2-b^2}{b^2} \quad \text{from (65).}$$

thus:

$$\cos \phi = \frac{2a_1a_2-b^2}{b^2} \tag{67}$$

The currents in the odd cavities may be found by eliminating V_p between equations (61) and (64). This gives:

$$I_p = \frac{-b}{2a_1} \left(I_n + I_{n+1} \right)$$

$$= \frac{-b}{2a_1} \frac{\cos (N-n-1)\,\phi + \cos (N-n)\,\phi}{\cos N\phi} I_o \qquad (68)$$

Thus currents in even and odd cavities are given by equations (66), and (68) respectively.

3.5 CURRENTS AT THE π-MODE IN A LOSSLESS STRUCTURE.

Because a periodic section of a doubly periodic structure consists essentially of two cavities, the phase shift per section, if the cavities were identical, would be equal to twice the phase shift of a singly periodic structure. The π-mode in a doubly periodic structure would be equivalent to the $\pi/2$-mode in a singly periodic structure. To compare with the results of Chapter (2), therefore, it is instructive to consider π-mode operation of the doubly periodic structure. In the lossless case, the currents in the even cavities are, from Equation (66) with $\psi = \pi$:

$$I_n = \cos n\pi \; I_o = (-1)^n \; I_o$$

the currents in the odd cavities are given by equation (68);

$$I_p = -\frac{b}{2a_1} \left(I_n + I_{n+1} \right) = -\frac{b}{2a_1} \left[(-1)^n + (-1)^{n+1} \right] = 0$$

This is only true if $a_1 \neq o$. For $\psi = \pi$, this means that $a_2 = o$.
(ie) Thus even cavities are energized and odd cavities are empty as was formed in Chapter 2.

3.6 CURRENTS AT THE π-MODE IN A LOSSY STRUCTURE.

If losses are introduced into the structure the phase shift becomes complex and is given by equation (35).

$$\phi = \psi - j\alpha$$

Where ψ and α are defined as before for one periodic section. With the aid of equation (36), equation (66) may be written as:

$$I_n = \frac{\cos (N-n)\psi \cosh (N-n)\alpha + j \sin (N-n)\psi \sinh (N-n)\alpha}{\cos N\psi \cosh N\alpha + j \sin N\psi \sinh N\alpha} I_o$$

Using π- mode operation the currents in the even cavities become:

$$I_n = \cos n\pi \frac{\cosh (N-n)\alpha}{\cosh N\alpha} I_o = (-1)^n \frac{\cosh (N-n)\alpha}{\cosh N\alpha} I_o \tag{69}$$

for odd cavities the currents are:

$$I_p = \frac{-b}{2a_1} \cos n\pi \frac{\cosh (N-n)\alpha - \cosh (N-n-1)\alpha}{\cosh N\alpha} I_o$$

$$= \frac{-b}{a_1} (-1)^n \frac{\sinh (N-n-\frac{1}{2})\alpha \sinh(\alpha/2)}{\cosh N\alpha} \tag{70}$$

These results are similar to those obtained for a singly periodic structure and reduce to them when $a_1=a_2$.

3.7 DISPERSION RELATION AND THE GROUP VELOCITY.

The dispersion relation is defined by equation (67). The phase shift is complex, in general, so that we must follow the same steps as in chapter 2 to separate the real and imaginary parts of equation (67). Substituting for a_1, a_2, and b from equation (59) into equation (67), gives:

$$\cos \phi = \frac{2\left(\frac{R_1}{2} + jWL_1 + \frac{1}{2jWC_1}\right)\left(\frac{R_2}{2} + jWL_2 + \frac{1}{2jWC_2}\right)}{-M^2W^2} - 1$$

$$= \left(\frac{2L_1L_2}{M^2} + \frac{1}{2M^2C_1C_2W^4} - \frac{R_1R_2}{2W^2M^2} - \frac{L_2C_2 + L_1C_1}{M^2W^2C_1C_2}\right) \tag{71}$$

$$+ j\left(\frac{R_1}{2W^3M^2C_2} + \frac{R_2}{2W^3M^2C_2} - \frac{L_2R_1 + L_1R_2}{M^2W}\right)$$

Equating the real and imaginary parts of equations (36) and equation (71) we have:

$$\cos\psi \cosh\alpha = \frac{2L_1L_2}{M^2} + \frac{1}{2M^2C_1C_2W^4} - \frac{R_1R_2}{2W^3M} - \frac{L_1C_1 + L_2C_2}{M^2C_1C_2W^2} - 1$$

$$\text{and} \quad \sin\psi \sinh\alpha = \frac{R_1}{2W^3M^2C_2} + \frac{R_2}{2W^3M^2C_1} - \frac{L_1R_2+L_2R_1}{M^2W} \tag{72}$$

If we define as before:

$$2L_1C_1 = W_1^{-2} \qquad Q_1 = \frac{2W_1L_1}{R_1}$$

$$2L_2C_2 = W_2^{-2} \qquad Q_2 = \frac{2W_2L_2}{R_2}$$

$$M = k\sqrt{L_1L_2}$$

Equation (72) may be written:

$$\cos\psi \cosh\alpha = \frac{2}{k^2} + \frac{2W_1^2W_2^2}{k^2W^4} - 2\frac{\left(W_1^2 + W_2^2\right)}{k^2W^2} - 2\frac{W_1W_2}{k^2Q_1Q_2W^2} - 1$$

$$\text{and} \quad \sin\psi \sinh\alpha = \frac{2W_1W_2^2}{k^2Q_1W^3} + \frac{2W_2W_1^2}{k^2Q_2W^3} - \frac{2W_2}{k^2Q_2W} - \frac{2W_1}{k^2Q_1W} \tag{73}$$

Thus, the dispersion relation for a doubly periodic structure is defined in terms of the resonant frequencies W_1, W_2, the coupling coefficient and the Q's of the cavities.

These results can now be applied to (a) a lossless structure and (b) a lossy structure.

(a) For a lossless structure $Q_1=Q_2 = \infty$ so that equation (73) reduces to:

$$\cos\psi = \frac{2}{k^2} + \frac{2W_1^2W_2^2}{k^2W^4} - 2\,\frac{\left(W_1^2+W_2^2\right)}{k^2W^2} - 1 \tag{74}$$

solving for the frequency in terms of the phase shift, we have:

$$W^4 + \left(\frac{W_1^2 + W_2^2}{k^2\left(\frac{1+\cos\psi}{2}\right)-1}\right)W^2 - \left(\frac{W_1^2W_2^2}{k^2\left(\frac{1+\cos\psi}{2}\right)-1}\right) = 0$$

$$\text{or} \qquad W^2 = -\frac{\left(W_1^2+W_2^2\right)}{2\left(k^2\left(\frac{1+\cos\psi}{2}\right)-1\right)} \pm \sqrt{\frac{W_1^2+W_2^2}{2\,k^2\left(\frac{1+\cos\psi}{2}\right)-1} + \frac{W_1^2W_2^2}{k^2\left(\frac{1+\cos\psi}{2}\right)-1}} \tag{75}$$

There are two branches of the dispersion curve. When the (-) sign is taken, $0 \leqslant \psi \leqslant \pi$, and when the (+) sign is taken, then $\pi \leqslant \psi \leqslant 2\pi$.

It is obvious that the curve defined above is discontinuous at ($\psi = \pi$), since from equation (75), $\psi = \pi$ corresponds to $W=W_1$ when the (+) sign is used, and $W=W_2$ when the (-) sign is used.

The group velocity is derived by differentiating equation (74), thus:

$$\frac{dW}{d\psi} = \frac{\sin\psi}{\frac{8W_1^2W_2^2}{k^2W^5} - \frac{4(W_1^2+W_2^2)}{k^2W^3}} \tag{76}$$

this shows that the group velocity at the π-mode is zero.

(b) For a lossy structure, the dispersion curve is given by equation (73). At $\psi = \pi$, the second equation in (73) indicates that:

$$0 = \frac{2W_1W_2}{k^2Q_1W^3} + \frac{2W_2W_1^2}{k^2Q_2W^3} - \frac{2W_2}{k^2Q_2W} - \frac{2W_1}{k^2Q_1W}$$

$$\text{so that} \quad W = \sqrt{\frac{W_1W_2^2Q_2 + W_2W_1^2Q_1}{W_2Q_1 + W_1Q_2}} \tag{77}$$

Thus the π-mode now occurs at only one value of the frequency. Examination of equation (73) shows that the dispersion curve is continuous at $\psi = \pi$, which indicates that the group velocity at the π-mode remains finite. The group velocity is obtained by differentiating equations (73) and eliminating $\frac{d\alpha}{dW}$.

$$\frac{dW}{d\psi} = \frac{\cos^2\psi \sinh^2\alpha + \sin^2\psi \cosh^2\alpha}{\left(\frac{2}{k^2W^2} - \frac{6W_1W_2}{k^2W^4}\right)\left(\frac{W_1}{Q_1} + \frac{W_2}{Q_2}\right)\cos\psi\sinh\alpha - \frac{4}{k^2W^3}\left\{W_1^2 + W_2^2 + \frac{W_1W_2}{Q_1Q_2} - \frac{2W_1^2W_2^2}{W^2}\right\}\sin\psi\cosh\alpha} \tag{78}$$

Equation (78) for ($\psi = \pi$) gives:

$$\left.\frac{dW}{d\psi}\right|_{\psi=\pi} = \frac{\left\{\left[\frac{2}{k^2} - 1 + \frac{2W_1^2W_2^2(W_2Q_1+W_1Q_2)^2}{k^2(W_1W_2^2Q_2+W_2W_1^2Q_1)^2} - \frac{2}{k^2}\left(W_1^2+W_2^2+\frac{W_1W_2}{Q_1Q_2}\right)\frac{(W_2Q_1+W_1Q_2)}{(W_1W_2^2Q_2+W_2W_1^2Q_1)}\right]^2 - 1\right\}^{\frac{1}{2}}}{\frac{2}{k^2}\left(\frac{W_1}{Q_1} + \frac{W_2}{Q_2}\right)\left(\frac{3W_1W_2(W_2Q_1+W_1Q_2)^2}{(W_1W_2^2Q_2+W_2W_1^2Q_1)^2} - \frac{W_1Q_2+W_2Q_1}{W_1W_2^2Q_2+W_2W_1^2Q_1}\right)} \tag{79}$$

This is the group velocity at the π-mode. Equation (79) shows that the group velocity is finite at the π-mode except when Q_1 or Q_2 are infinite. The group velocity is then zero. It is interesting to note that this will occur at W_1, if $Q_1=\infty$ or at W_2, if $Q_2=\infty$.

The dispersion curve is plotted in Fig. 4 for a typical structure with and without lossess to illustrate the effect of losses on the stop band.

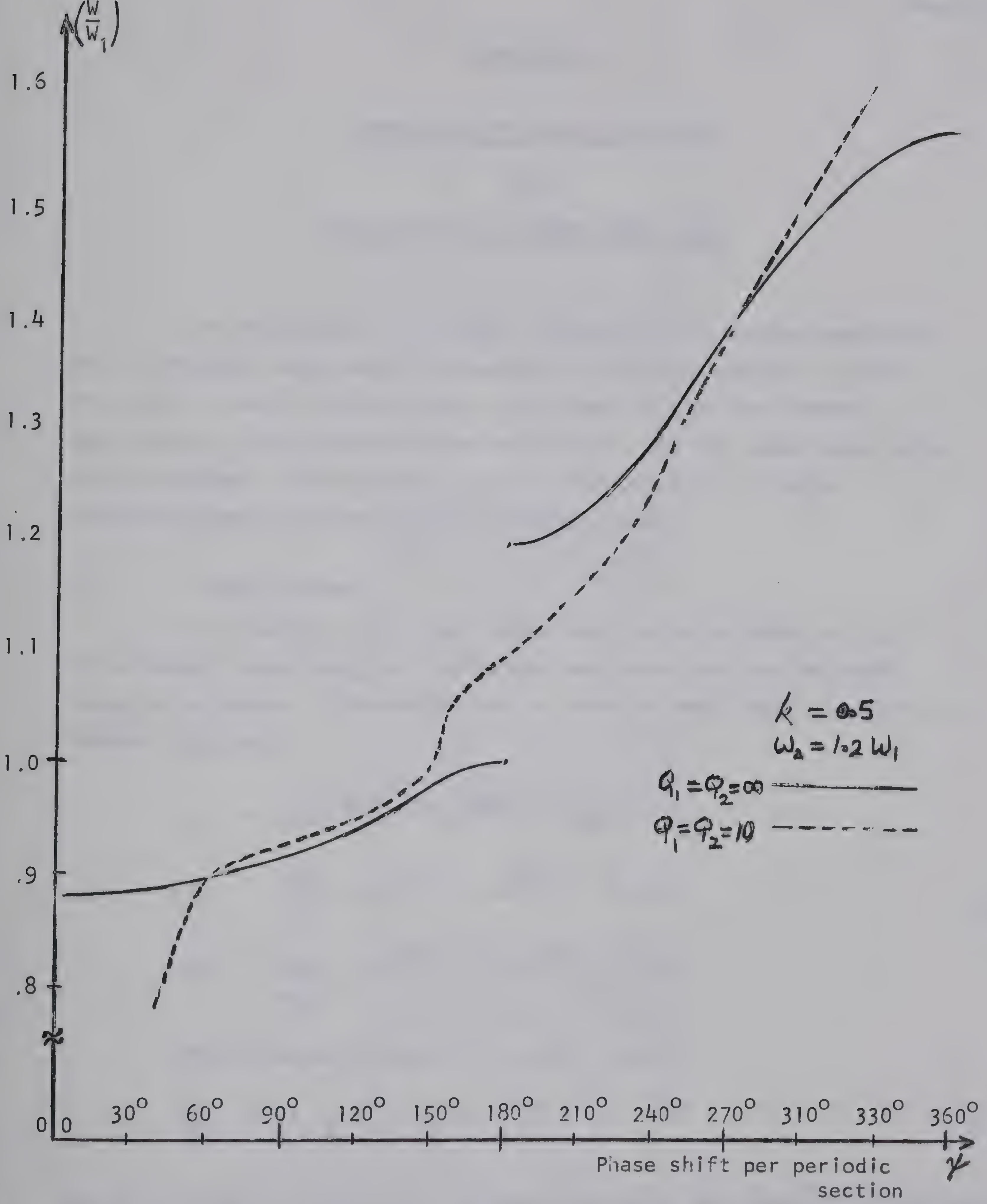


Fig. 14: The dispersion curve with and without losses for a doubly periodic structure.

CHAPTER IV

COMPARISON OF COUPLED CAVITIES TO A DIELECTRICALLY LOADED WAVE GUIDE

An equivalent circuit for a dielectrically loaded wave guide may be obtained using Maxwell equations. The field pattern in this structure is easily derived due to the simplicity of the boundary conditions. Thus an exact equivalent circuit for the loaded wave guide can be obtained. The analysis will be restricted to a circular cylindrical wave guide operated in the TMo_1 mode.

4.1 FIELD PATTERN

A circular cylindrical loaded wave guide is shown in Fig. 15. Using Maxwell equations, the fields that may exist for the TMo_1-mode assuming no losses (the definition is given in many books, see Ref. 9, Chapter 8.) are:

$$E_z = (Ge^{-j\beta_1 z} + He^{j\beta_1 z})\ J_o(\chi r)$$

$$E_r = \frac{j\beta_1}{\chi}(Ge^{-j\beta_1 z} - He^{j\beta_1 z})\ J_1(\chi r) \tag{80}$$

$$H\phi = \frac{jW\varepsilon_1}{\chi}(Ge^{-j\beta_1 z} + He^{j\beta_1 z})\ J_1(\chi r)$$

For a region of permittivity ε_1, where:

$$\beta_1^2 = W^2 u\,\varepsilon_1 - \chi^2 \tag{81}$$

Now for a guide of raduis 'a'. E_z must vanish at r=a, hence it is necessary that:

$$J_o(\chi a) = 0$$

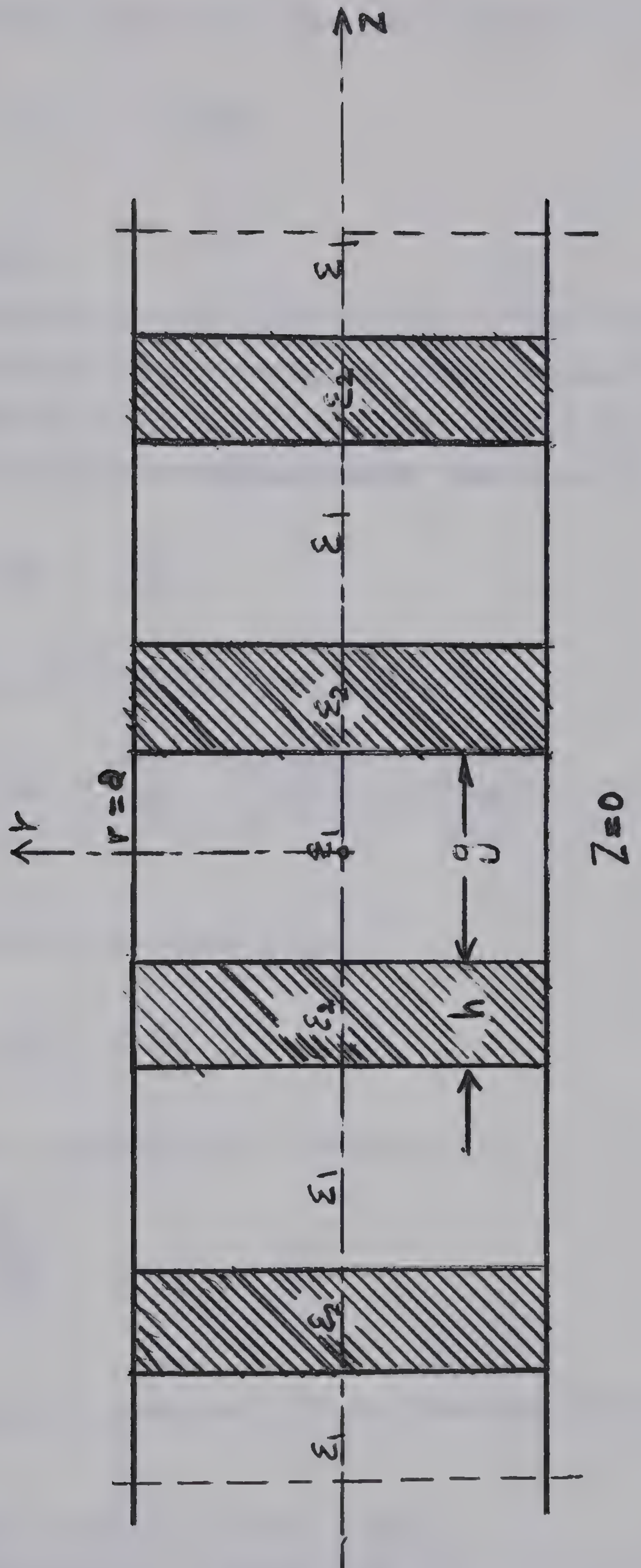


Fig. 15: A circular cylindrical wave guide loaded with dielectric disks.

For Eo_1-mode the first root of the Bessel function is taken so that:

$$\chi a = S_1 = 2.4048...$$

4.2 IMPEDANCE

The impedance concept enables one to take into account the effects of the boundary conditions which exist between two regions of different permittivities. In a circular cylindrical wave guide, using cylindrical coordinates, the ~~characteristic~~ impedance at any point is:

$$Z_T\ (z,r,\phi) = \frac{Er}{H\phi} \tag{82}$$

Substituting for E_r, $H\phi$ from Equation (77) gives:

$$Z_T\ (z,r,\phi) = \frac{\beta_1}{W\varepsilon_1} \frac{Ge^{-j\beta_1 z} - He^{j\beta_1 z}}{Ge^{-j\beta_1 z} + He^{j\beta_1 z}} \tag{83}$$

which is a function of (z) alone (ie):

$$Z_T(z,r,\phi) = Z_T(z)$$

The characteristic impedance for a TM mode is:

$$Z_1 = \frac{\beta_1}{W\varepsilon_1} \tag{84}$$

So that substituting in equation (79) and rearranging we have:

$$Z_T(z) = Z_1 \frac{(G-H)\cos\beta_1 z - j(G+H)\sin\beta_1 z}{(G+H)\cos\beta_1 z - j(G-H)\sin\beta_1 z} \tag{85}$$

A block diagram of a wave guide of length "L" characterized by ε_1 is shown in Fig.15 below:

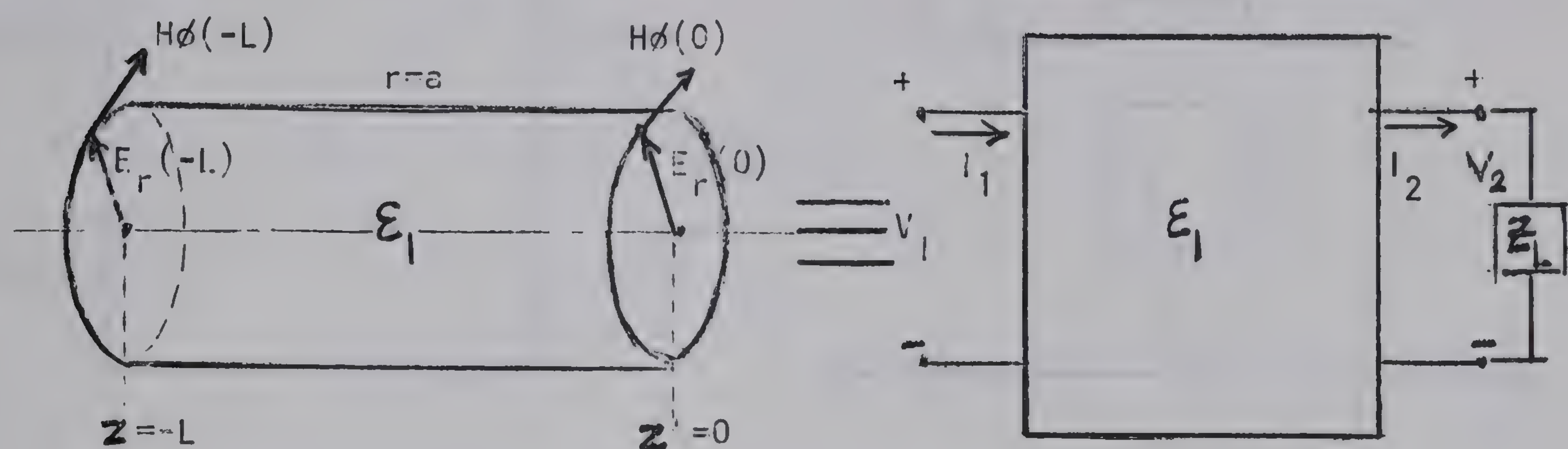


Fig. 16: Block diagram of wave guide piece E_r, is equivalent to V and Hø is equivalent to I.

The relation between the electric and magnetic field or the voltage and current depends on the value of the impedance at z=o. Thus, if the wave guide is terminated in an impedance Z_L, equation (85) gives:

$$Z_T(o) = Z_L = Z_1\frac{(G-H)}{(G+H)} \tag{86}$$

From equations (86) and (85):

$$Z_T(z) = Z_i\left(\frac{Z_L\cos\beta_1 z - jZ_1\sin\beta_1 z}{Z_1\cos\beta_1 z - jZ_L\sin\beta_1 z}\right) \tag{87}$$

which is the well-known transmission line equation.

4.3 EQUIVALENT CIRCUIT.

An equivalent circuit for a wave guide section, as indicated in Fig. 17, may be drawn using equation (87).

Since the section is symmetric with respect to the z axis, it follows that only two impedances have to be determined.

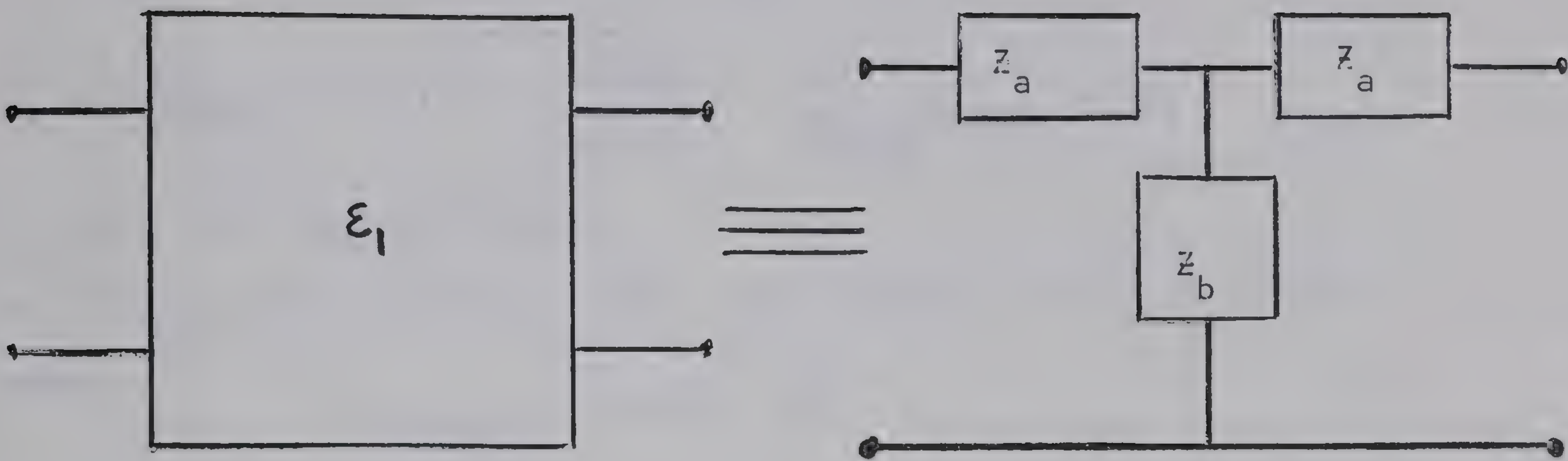


Fig. 17: An equivalent (T) circuit of the block diagram in Fig. 16.

The impedance at the input ($z=-L$) is:

$$Z_T(-L) = Z_{in} = Z_1\left(\frac{Z_L \cos\beta_1 L + jZ_1 \sin\beta_1 L}{Z_1 \cos\beta_1 L + jZ_L \sin\beta_1 L}\right) \text{ from equation (87).}$$

If we let $Z_L = 0$ then:

$$Z_{in} = jZ_1 \tan\beta_1 L$$

and if $Z_L = \infty$, then:

$$Z_{in} = -jZ_1 \cot\beta_1 L$$

From Fig. 17 for short and open circuits we must have:

$$Z_a + \frac{Z_a Z_b}{Z_a + Z_b} = jZ_1 \tan\beta_1 L = jZ_1 \tan\theta_1$$

$$\text{and} \qquad Z_a + Z_b = -j Z_1 \cot\beta_1 L = -jZ_1 \cot\theta_1 \tag{88}$$

Where $\theta_1 = \beta_1 L$

solving for Z_a and Z_b in equation (84) we have:

$$Z_a = \frac{Z_1}{j \tan \theta_1} \pm Z_1 \frac{\sqrt{1 + \tan^2 \theta_1}}{j \tan \theta_1} = \frac{Z_1}{j \sin \theta_1} (\cos \theta_1 \pm 1)$$

Z_a must tend to zero as $\theta_1 \rightarrow 0$

so that we must take the (-) sign. This implies that Z_a is inductive

Hence:

$$Z_a = \frac{Z_1}{j \sin \theta_1} (\cos \theta_1 - 1) \tag{89}$$

and $$Z_b = \frac{Z_1}{j \sin \theta_1}$$

From Fig. 17:

$$V_1 = (Z_a + Z_b) I_1 - Z_b I_2$$

$$V_2 = -(Z_a + Z_b) I_2 + Z_b I_1$$

Substituting for Z_a and Z_b from equation (89) and solving for V_2, I_2 in terms of V_1, I_1 one gets:

$$\begin{pmatrix} V_2 \\ I_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & -jZ_1 \sin \theta_1 \\ \frac{-j \sin \theta_1}{Z_1} & \cos \theta_1 \end{pmatrix} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} \tag{90}$$

If two sections of a wave guide with different permittivities are joined together, then a block diagram may be drawn by connecting the output of one section of permittivity (ε_1) to the input of the other section of permittivity (ε_2). Fig. 18 below shows a block diagram of two sections characterized by (ε_1) and (ε_2). The length of section (1) is ($\frac{g}{2}$) while that of section (2) is ($\frac{h}{2}$).

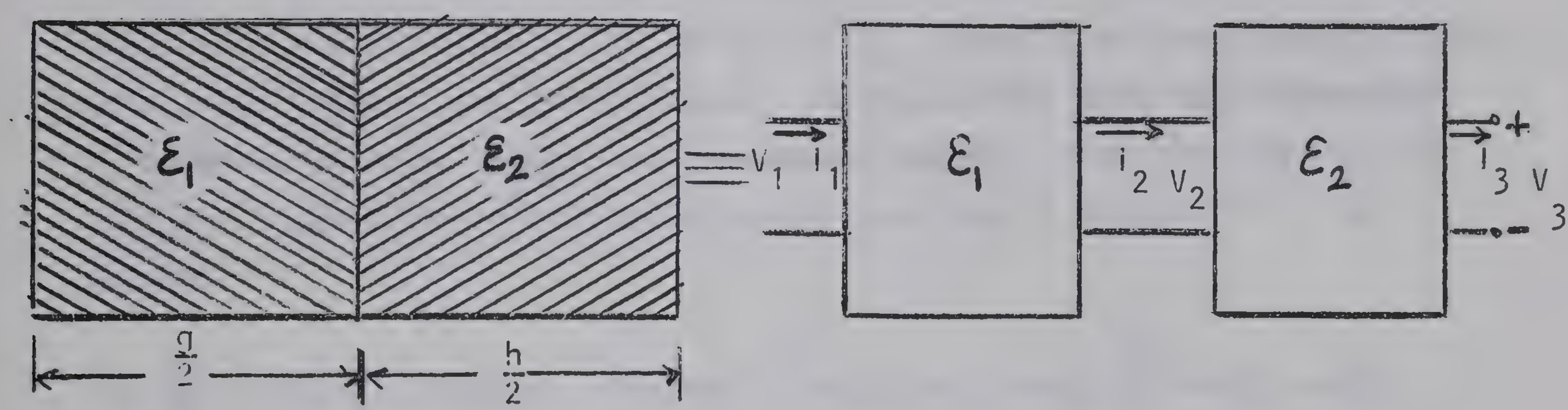


Fig. 18: A block diagram of two wave guide sections of different permittivities.

From Fig. 18 and equation (90), it follows that:

$$\begin{pmatrix} V_3 \\ I_3 \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & -jZ_2\sin\theta_2 \\ \frac{-j\sin\theta_2}{Z_2} & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & -jZ_1\sin\theta_1 \\ \frac{-j\sin\theta_1}{Z_1} & \cos\theta_1 \end{pmatrix} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

Where $\theta_2 = h/2\,\beta_2$ and $\theta_1 = g/2\,\beta_1$.

Using matrix multiplication, we have:

$$\begin{pmatrix} V_3 \\ I_3 \end{pmatrix} = \begin{pmatrix} \cos\theta_2 \cos\theta_1 - \frac{Z_2}{Z_1} \sin\theta_2 \sin\theta_1 & -j\left(Z_1 \sin\theta_1 \cos\theta_2 + Z_2 \sin\theta_2 \cos\theta_1\right) \\ -j\left(\frac{\sin\theta_2 \cos\theta_1}{Z_2} + \frac{\sin\theta_1 \cos\theta_2}{Z_1}\right) & \cos\theta_2 \cos\theta_1 - \frac{Z_1}{Z_2} \sin\theta_2 \sin\theta_1 \end{pmatrix} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} \tag{91}$$

This is the transformation matrix equivalent to T_1 in Chapter III. For a periodic section of a dielectrically loaded wave-guide, we must take the half section shown in Fig. 18, plus its reverse. The matrix T_2 for the reversed half section is similar to T_1, except that the elements of the principle diagonal are interchanged. ~~It is obvious from the discussion in Chapter III that (T_2T_1) is a symmetric matrix.~~ From the elements of T_2T_1, the dispersion relationship can be found as in Chapter III. We obtain:

$$\cos\psi = \left(\cos\theta_2 \cos\theta_1 - \frac{Z_2}{Z_1} \sin\theta_2 \sin\theta_1\right)\left(\cos\theta_2 \cos\theta_1 - \frac{Z_1}{Z_2} \sin\theta_2 \sin\theta_1\right)$$

$$- \left(Z_1 \sin\theta_1 \cos\theta_2 + Z_2 \sin\theta_2 \cos\theta_1\right)\left(\frac{\sin\theta_2 \cos\theta_1}{Z_2} + \frac{\sin\theta_1 \cos\theta_2}{Z_1}\right)$$

Simplifying, one gets:

$$\cos\psi = \cos 2\theta_2 \cos 2\theta_1 - \frac{1}{2}\left(\frac{Z_1}{Z_2} + \frac{Z_2}{Z_1}\right) \sin 2\theta_1 \sin 2\theta_2 \tag{92}$$

This is the dispersion relation* for a periodically loaded wave guide structure. A typical dispersion curve is shown in Fig. 19.

* SEE: Reference 10

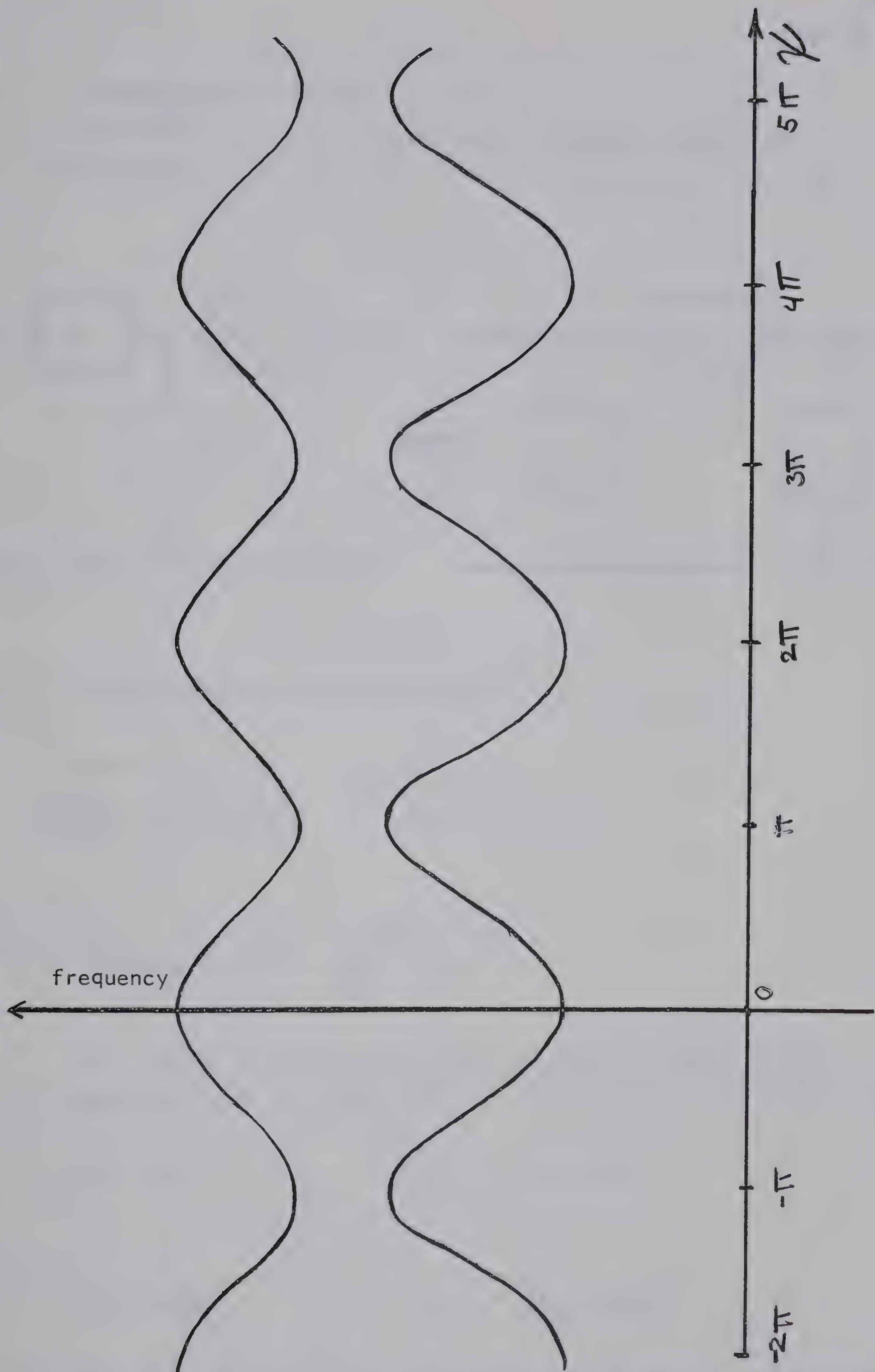


Fig. 19: Part of first and second pass bands in general dielectrically loaded structure.

4.4 COMPARISON WITH COUPLED CAVITIES.

The "Y" equivalent circuit of a wave guide section in Fig. 17 may be converted into "Δ". This is done below in Fig. 20:

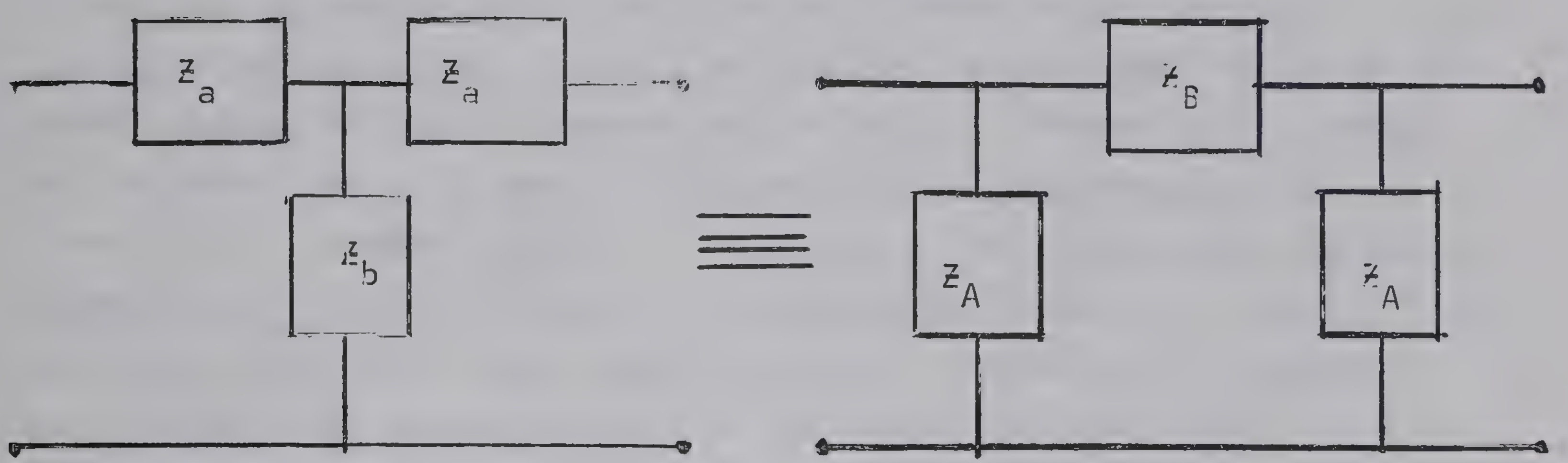


Fig. 20: "Δ" Equivalent circuit of the "Y"

Where:

$$Z_A = \frac{Z_a Z_b + Z_a^2 + Z_a Z_b}{Z_a} = Z_a + 2 Z_b$$

$$\text{and } Z_B = \frac{Z_a Z_b + Z_a^2 + Z_a Z_b}{Z_b} = \frac{Z_a}{Z_b}(2Z_b + Z_a)$$

For region (1) whose permittivity is ε_1, and whose length is "g", and for region (2) whose length is "h", the Delta Impedences are:

$$Z_{A_1} = -j Z_1 \cot \theta_1 \qquad Z_{A_2} = -jZ_2 \cot \theta_2$$

$$Z_{B_1} = j Z_1 \sin 2\theta_1 \qquad Z_{B_2} = jZ_2 \sin 2\theta_2$$

An equivalent circuit of two sections of a wave guide characterized by (ε_1) and (ε_2), which are part of a periodic structure, are shown in Fig. 21, (a) and (c).

The parallel impedances in (b) have been added to give the total equivalent circuit in (c). This may be compared with the equivalent circuit for a coupled cavity chain which is shown in (d) and (e). It is seen that the equivalent circuits are similar so that each region of the loaded wave guide can be compared with a cavity. Theoretically, values can be chosen for L, C, and M, to agree with the impedances for a dielectrically loaded structure. In practice, the values obtained may not be physically possible since M is limited by the values of L_1 and L_2. It should be noted that in the loaded structure, the value of M depends upon the physical characteristics of the coupled sections and is not an independent quantity.

The resonant frequency of a region is obtained by equating the sum of the impedances round a closed loop to zero.

Thus from Fig. 21 (c), for region (1), we have:

$$jZ_1 \sin 2\theta_1 - j_2 \frac{Z_1 Z_2 \cot\theta_1 \cot\theta_2}{Z_1 \cot\theta_1 + Z_2 \cot\theta_2} = 0$$

Simplifying, we have:

$$\tan\theta_1 \tan\theta_2 = \frac{Z_2}{Z_1} \tag{93}$$

The solution of this equation defines a frequency W_1.

Similarly, for region (2), we have:

$$\tan\theta_1 \tan\theta_2 = \frac{Z_1}{Z_2} \tag{94}$$

Which defines W_2.

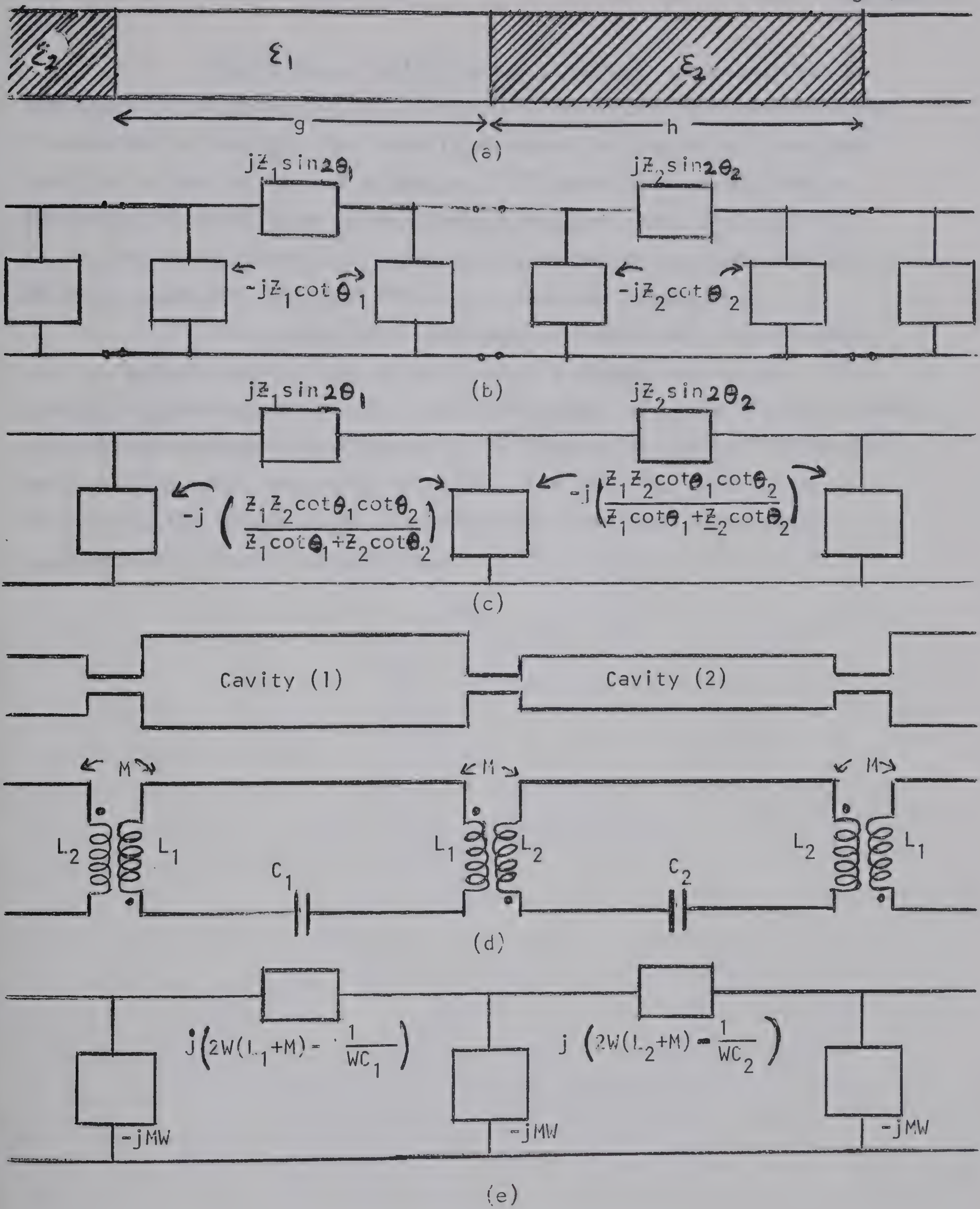


Fig. 21: The equivalent circuits of the cavities and the dielectrically loaded wave guide. (a) to (c) is for the wave guide, and (d) to (e) is for the cavities.

The dielectrically loaded structure may therefore be considered as a doubly periodic structure whose cavities have resonant frequencies W_1 and W_2. The results obtained in Chapter III indicate that for either of these frequencies, π-mode will occur. For a confuence at the π-mode we must have $W_1=W_2$ and hence $Z_1=Z_2$. This is in precise agreement with the results of Walker and West (3), who obtained equations (93) and (94) by a different method.

The theory of coupled cavity chains may therefore be used to predict some of the properties of a loaded wave guide. The analogy is, however, not exact. Since the values of L and C that represent the wave guide depend on frequency, the results of Chapter III cannot be used to predict the group velocity. For the same reason, the dielectrically loaded guide has infinitely many pass bands while a coupled cavity chain has only two.

CONCLUSION

In this work, a simple model for a chain of coupled cavities has been investigated. The parameters that describe the model are assumed to vary slowly with frequency so that they can be taken as constant in the vicinity of the operating frequency. This approach holds true, only as long as the coupling mechanism introduces a small perturbation on the field in each cavity.

An investigation of the singly periodic structure shows that it has only one pass band. The phase shift varies between zero and π and at both of these limits the group velocity is zero. When operated at the $\pi/2$-mode, with a short circuit termination, the currents in the even cavities have the same magnitude while the currents in the odd cavities are zero. When the structure is arranged as in Fig. 1 (a), the even cavities simulate a π-mode operation.

Losses introduced into the structure will affect the currents in the cavities. Odd cavities will have a current proportional to the losses, while the amplitude of the currents in even cavities are only slightly affected. In general, the dispersion curve will be steeper when losses are introduced; in particular, the dispersion curve will be most affected at the end points, zero and π.

The doubly periodic structure investigated consists of two types of cavities. These are identified by their resonant frequencies, W_1 and W_2. It was found that for a lossless structure there are two pass bands covering the range 0 to π and π to 2π. The stop band at ($\psi = \pi$) covers the range of frequency between W_1 and W_2. If the cavities are made identical, the stop band disappears and the doubly periodic structure reduces to singly periodic structure with a $\pi/2$

phase shift per cavity. Thus π-mode operation in a doubly periodic structure is equivalent to $\pi/2$-mode operation in a singly periodic structure. Losses introduced into the doubly periodic structure will close the stop band and the π-mode will now occur at only one frequency. This frequency lies between W_1 and W_2, and is closer to the resonant frequency of the more lightly loaded cavity. In particular, if the coupling cavities are lossless, the π-mode frequency will be the resonant frequency of the coupling cavities even if the accelerating cavities are only slightly loaded.

Nagle (11) has shown that in a lossless doubly periodic structure $\psi = \pi$ will occur at W_1 and W_2, a result which has been confirmed. However, if the cavities are lossy, the π - mode can only be obtained at one frequency, which will be near to the resonant frequency of the lightly loaded coupling cavities.

Effects of losses on the rest of the dispersion curve are similar to the singly periodic case. The currents in the cavities are similar to those in the singly periodic case. When losses are introduced into either the coupling cavities or the accelerating cavities, then a current is produced in the odd cavities.

An exact equivalent circuit of a dielectrically loaded wave guide was found. A comparison between the equivalent circuit of coupled cavities and a dielectrically loaded wave guide showed that the air region and the dielectric regions in the wave guide may be considered as the accelerating and coupling cavities respectively.

Although many of the properties of the two systems are similar, the analogy is not exact. As already stated, the coupled circuit model for the cavity chain only applies when the coupling is small. The field pattern in each cavity is then independent of the field in adjacent cavities. In the dielectrically loaded wave guide the field in each region can only be determined with reference to the whole structure.

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